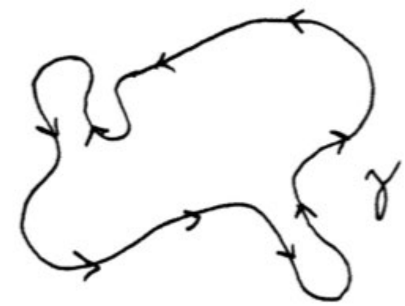
Notice something weird about this post? That's right, it's all handwritten, because this post is going to have a lot of diagrams. <u>AND</u> this is a <u>very special</u> post, in which I will present the magnificent <u>RESIDUE THEOREM</u>, and reveal its awesome ability to allow one to evaluate many a difficult integral and infinite series with ease.

I'll start with Cauchy's Integral Theorem, with which I assume the reader is already somewhat familiar. It states the following:



$$\oint_\gamma f(z)\,dz = 0$$

> If $f(z)$ is a holomorphic function at all points in and on the simple closed curve $\gamma \subset \mathbb{C}$, then
>
> $$\oint_\gamma f(z)\,dz = 0$$

This is itself an interesting result, derived readily from the Cauchy-Riemann equations and Green's Theorem. However, the truly magnificent results that spring from this theorem do so when it <u>doesn't</u> <u>work</u>.

Let me explain. This theorem only tells us that the integral

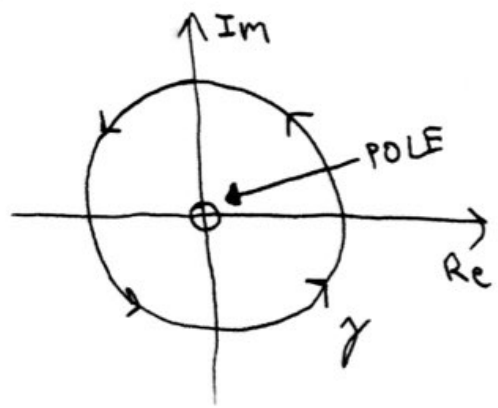
$$\oint_\gamma f(z)\, dz.$$

vanishes if f is holomorphic at all points in and on γ. But what if f isn't "well behaved?" For example, what about the function

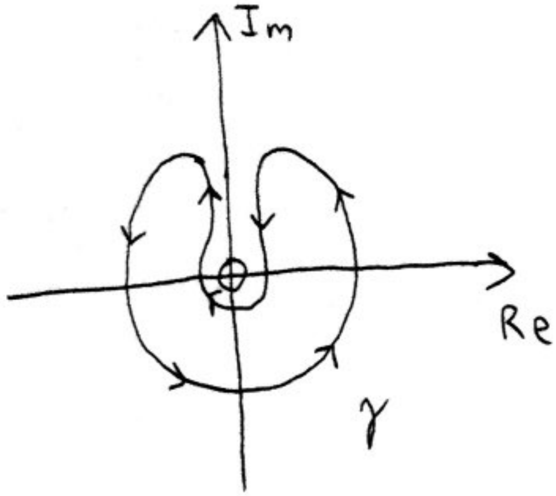
$$f(z) = \frac{e^{iz}}{z}$$

which has a pole at $z=0$?

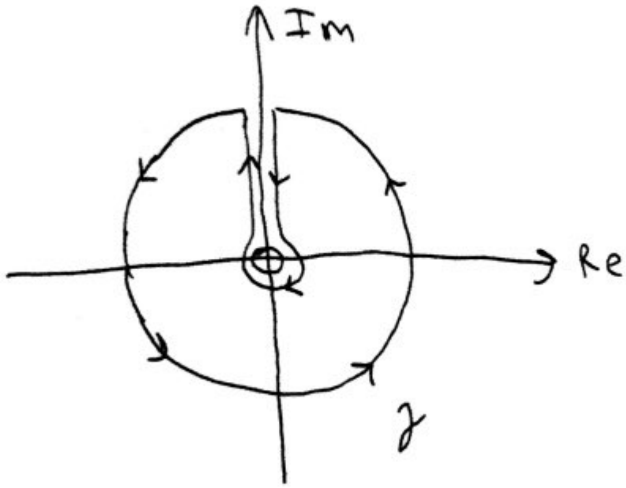
Here's the trick: we can change the ~~[illegible]~~ contour γ about which we integrate. Suppose, for example, that we want to integrate $f(z)$ about the unit circle in the complex plane, so that γ is the circle $|z|=1$. The integral isn't necessarily zero, since f explodes at $z=0$.



Okay, so Cauchy's Integral theorem won't do this one for us. But it can do the integral around γ if we change it a little bit to omit the trouble spot at $z=0$:

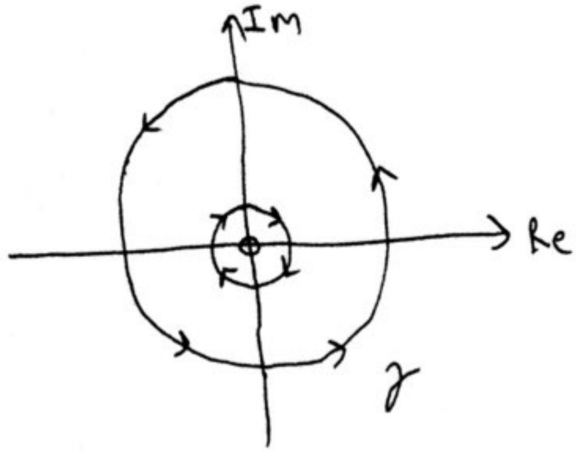


The integral around this contour, we know, is <u>zero</u>, but it's no unit circle. Perhaps we can keep the pole out of our contour but make it look a little more like the contour we want:



It looks a lot like the unit circle, but we've cut a little key-hole-shaped dent in it. We can make it look even more like the unit circle by making our keyhole tighter:

Notice that, as we squeeze the keyhole tighter and tighter, the two straight segments become practically identical, but face in opposite directions, essentially cancelling each other. Thus, we can pretty much ignore them, and what we end up with is something that looks like this:



Two concentric circles, oriented in opposite directions! Remember that the integral around this union of two contours is still equal to zero. If we let ε be the radius of the inner circle, we end up with

$$\int_{|z|=1} f(z)\,dz - \int_{|z|=\varepsilon} f(z)\,dz = 0$$

By parametrizing the integrals, this becomes

$$\int_0^{2\pi} \frac{e^{ie^{i\theta}}}{e^{i\theta}} \cdot ie^{i\theta}\,d\theta = \int_0^{2\pi} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta}\,d\theta$$

or

$$\int_0^{2\pi} e^{ie^{i\theta}}\,d\theta = \int_0^{2\pi} e^{i\varepsilon e^{i\theta}}\,d\theta$$

Now let's squeeze our keyhole even more by letting the radius ε of the inner circle become arbitrarily small, approaching zero. By taking the limit as $\varepsilon \to 0$ of the previous equation, we have

$$\int_0^{2\pi} e^{ie^{i\theta}}\, d\theta = \int_0^{2\pi} e^0\, d\theta$$

or

$$\int_0^{2\pi} e^{ie^{i\theta}}\, d\theta = 2\pi$$

Now, if we simplify the left side a bit, we have

$$\int_0^{2\pi} e^{-\sin\theta} \cos(\cos\theta)\, d\theta + \int_0^{2\pi} i e^{-\sin\theta} \sin(\cos\theta)\, d\theta = 2\pi$$

By equating real parts, we have

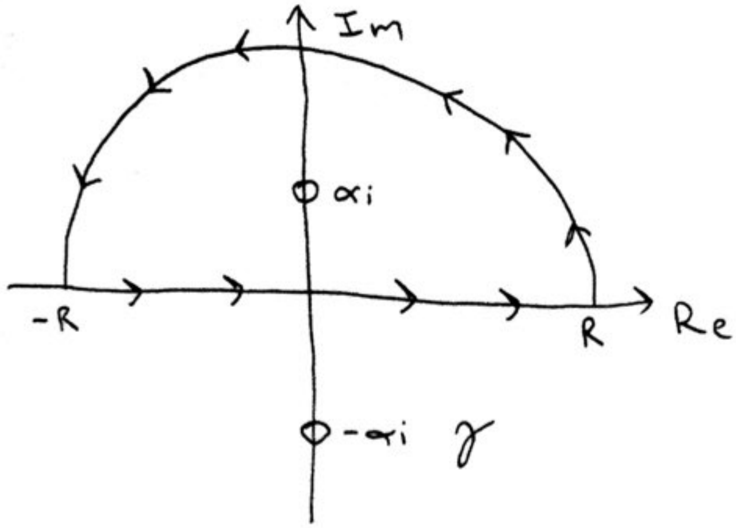
$$\boxed{\int_0^{2\pi} e^{-\sin(\theta)} \cos(\cos\theta)\, d\theta = 2\pi}$$

You may notice that this integral is similar to one evaluated in an earlier post using the Taylor series of e^x. It may not seem so impressive since you've already seen another potential method of evaluating it, but my next example isn't so easily done using elementary means.

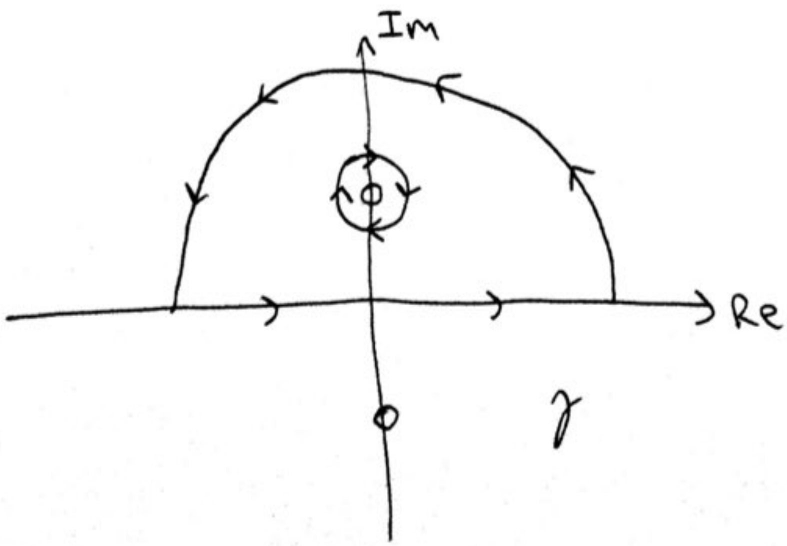
Consider now the function

$$F(z) = \frac{e^{iz}}{z^2 + \alpha^2}$$

where α is a positive real number. Define γ to be a semicircular contour with radius R, as depicted below:



As shown above, this contour also contains a "trouble spot": A pole of f at $z = \alpha i$. But we can easily cut it out using the keyhole method from before:



Now let's parametrize. The semicircular part of our contour is given by

$$\int_{-R}^{R} \frac{e^{iz}}{z^2+\alpha^2}\,dz + \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{R^2e^{2i\theta}+\alpha^2}\cdot iRe^{i\theta}\,d\theta$$

and the circular "keyhole" is given by

$$-\int_0^{2\pi} \frac{e^{i\varepsilon e^{i\theta}-\alpha}}{(\varepsilon e^{i\theta}+\alpha i)^2+\alpha^2}\cdot i\varepsilon e^{i\theta}\,d\theta$$

Note that the first integral refers to the line-segment portion of the semicircle, and the second refers to the arc. As we did last time, let's let ε approach zero. The first two integrals remain constant, while the third becomes

$$-\lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\cdot e^{i\varepsilon e^{i\theta}-\alpha}}{(\varepsilon e^{i\theta}+\alpha i)^2+\alpha^2}\,d\theta$$

$$= -i\int_0^{2\pi} \lim_{\varepsilon\to 0} \frac{\varepsilon e^{i\theta}\cdot e^{i\varepsilon e^{i\theta}-\alpha}}{(\varepsilon e^{i\theta}+\alpha i)^2+\alpha^2}\,d\theta$$

$$= -i\int_0^{2\pi} e^{-\alpha}\lim_{\varepsilon\to 0} \frac{\varepsilon e^{i\theta}}{(\varepsilon e^{i\theta}+\alpha i)^2+\alpha^2}\,d\theta$$

$$= -i\int_0^{2\pi} e^{-\alpha}\lim_{\varepsilon\to 0} \frac{e^{i\theta}}{2e^{i\theta}(\varepsilon e^{i\theta}+\alpha i)}\,d\theta$$

$$= -i\int_0^{2\pi} e^{-\alpha}\cdot\frac{1}{2\alpha i}\,d\theta$$

$$= -\frac{\pi e^{-\alpha}}{\alpha}$$

This tells us that

$$\int_{-R}^{R} \frac{e^{iz}}{z^2+\alpha^2}\,dz + \int_0^{\pi} \frac{iRe^{i\theta}\, e^{iRe^{i\theta}}}{R^2 e^{2i\theta} + \alpha^2}\,d\theta = \frac{\pi e^{-\alpha}}{\alpha}$$

... which is really gross and not particularly interesting. But just as we squished our keyhole and let $\varepsilon \to 0$, we can stretch out our semicircle and make it arbitrarily big, letting $R \to \infty$. If we do this, the first integral becomes

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+\alpha^2}\,dz$$

The second integral vanishes nicely as $R \to \infty$, since the expression

$$\frac{iRe^{i\theta}\, e^{iRe^{i\theta}}}{R^2 e^{2i\theta} + \alpha^2}$$

approaches zero as $R \to \infty$, as long as $0 < \theta < \pi$, which is guaranteed by the integral's bounds. Thus, we have

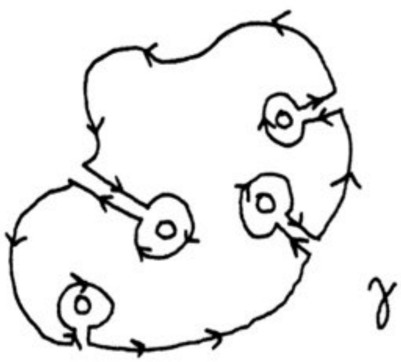
$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+\alpha^2}\,dz = \frac{\pi e^{-\alpha}}{\alpha}$$

or, by equating real parts,

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos(z)}{z^2+\alpha^2} = \frac{\pi e^{-\alpha}}{\alpha}}$$

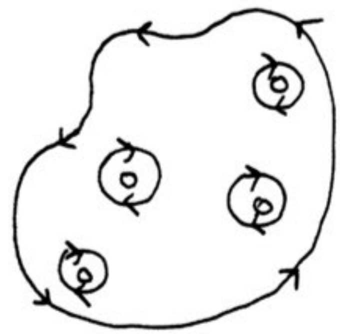
You may find that this monster integral is much more difficult to evaluate using other methods.

We can generalize this little "keyhole" trick that we've been using. Suppose that we have a function $f(z)$ with poles at $z_0, z_1, z_2, \ldots, z_k$, and a contour γ. We can cut out each pole using $k+1$ keyholes:



γ

As we shrink all of the keyholes to circles oriented clockwise about each pole, each with radius ε, we'll have something like this:



γ

The integral of f about the circle surrounding the pole z_j will be given by

$$-\int_0^{2\pi} f(z_j + \varepsilon e^{i\theta}) \cdot i\varepsilon e^{i\theta} \, d\theta$$

Adding the integral about the big contour to each of the integrals about the circles will give us zero by Cauchy's Integral Theorem, since we will have "cut out" each pole. Thus:

$$\oint_\gamma f(z)\,dz - \sum_j \int_0^{2\pi} f(z_j + \varepsilon e^{i\theta}) \cdot i\varepsilon e^{i\theta}\, d\theta = 0$$

or

$$\oint_\gamma f(z)\,dz = \sum_j \int_0^{2\pi} f(z_j + \varepsilon e^{i\theta}) \cdot i\varepsilon e^{i\theta}\, d\theta$$

The real question is this: what happens to the integral

$$\int_0^{2\pi} f(z_j + \varepsilon e^{i\theta}) \cdot i\varepsilon e^{i\theta}\, d\theta$$

when we let $\varepsilon \to 0$ as we did before?

To solve this, we must expand f into its Laurent Series about $z = z_j$:

$$f(z) = \cdots + \frac{a_{-3}}{(z-z_j)^3} + \frac{a_{-2}}{(z-z_j)^2} + \frac{a_{-1}}{z-z_j} + a_0 + a_1(z-z_j) + \cdots$$

This turns the integral in consideration into

$$\int_0^{2\pi} i\varepsilon e^{i\theta} \sum_{n=-\infty}^{\infty} a_n (\varepsilon e^{i\theta})^n\, d\theta$$

or

$$\sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} i\varepsilon e^{i\theta} (\varepsilon e^{i\theta})^n\, d\theta$$

$$= \sum_{n=-\infty}^{\infty} i\varepsilon^{n+1} a_n \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

Now notice that

$$\int_0^{2\pi} e^{i(n+1)\theta} \, d\theta$$

is equal to zero unless $n = -1$, in which case it is equal to 2π. Thus, all terms but the $n=-1$ term of the series vanish, leaving just

$$2\pi i a_{-1}$$

Now we make a small assumption: we assume that at $z = z_j$, f has a pole of <u>finite</u> <u>order</u>, or that z_j is not an <u>essential</u> singularity. This means that, for some m, $a_n = 0$ for all $n < -m$, or that f can be written in the form

$$f(z) = \frac{a_m}{(z-z_j)^m} + \ldots + \frac{a_{-1}}{z-z_j} + a_0 + a_1(z-z_j) + \ldots$$

An example of a function with a essential singularity (of infinite order) is $e^{1/z}$ (at $z=0$). If f <u>can</u> be written in this form, however, then we have

$$(z-z_j)^m f(z) = a_m + a_{1-m}(z-z_j) + \ldots + a_{-1}(z-z_j)^{m-1} + a_0(z-z_j)^m + \ldots$$

and, by differentiating both sides $(m-1)$ times,

$$\frac{d^{m-1}}{dz^{m-1}}(z-z_j)^m f(z) = (m-1)!\, a_{-1} + m!\, a_0 (z-z_j) + \ldots$$

By evaluating this at $z = z_j$, we have

$$\left.\frac{d^{m-1}}{dz^{m-1}}(z-z_j)^m f(z)\right|_{z=z_j} = (m-1)!\, a_{-1}$$

or

$$\boxed{a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_j)^m f(z) \Big|_{z=z_j}}$$

This is called the <u>residue</u> of f at $z=z_j$, where m is the <u>order</u> of the pole of f at $z=z_j$. It is usually denoted

$$\operatorname*{Res}_{z=z_j} f(z)$$

Earlier, we evaluated each circular integral to be

$$2\pi i a_{-1}$$

or

$$2\pi i \operatorname*{Res}_{z=z_j} f(z)$$

which tells us that

$$\boxed{\oint_\gamma f(z)\, dz = 2\pi i \sum_j \operatorname*{Res}_{z=z_j} f(z)}$$

This is the <u>Residue Theorem</u>! To demonstrate its awesome power, I will now show how to use it to evaluate an infinite series rather than an integral:

$$\sum_{n=0}^{\infty} \frac{1}{n^2+\alpha^2}$$

For my final example, I ask you to consider the function

$$f(z) = \frac{\pi \cot(\pi z)}{z^2 + \alpha^2}$$

around the contour γ, defined as the curve $|z| = R$, a circle centered at $z = 0$. The poles of f occur at $z = \alpha i$, $z = -\alpha i$, and any real integer value of z. The residues of f at its poles of $\pm\alpha i$ are given as follows:

$$\begin{aligned}
\operatorname*{Res}_{z = \pm\alpha i} f(z) &= \lim_{z \to \pm\alpha i} \frac{\pi(z \mp \alpha i)\cot(\pi z)}{z^2 + \alpha^2} \\
&= \lim_{z \to \pm\alpha i} \frac{\pi \cot(\pi z)}{z \pm \alpha i} \\
&= \frac{\pi \cot(\pm \pi \alpha i)}{\pm 2\alpha i} \\
&= -\frac{\pi \coth(\pi \alpha)}{2\alpha}
\end{aligned}$$

We evaluated this directly from the definition of a residue, noting that each pole $z = \pm\alpha i$ is of order 1. Because each pole of the cotangent function is also of order 1, we can calculate its residues at $z = n \in \mathbb{Z}$ similarly:

$$\operatorname*{Res}_{z=n} f(z) = \lim_{z \to n} \frac{\pi(z-n)\cot(\pi z)}{z^2+\alpha^2}$$

$$= \frac{1}{n^2+\alpha^2} \lim_{z \to n} \frac{\pi(z-n)}{\tan(\pi z)}$$

Now I use L'Hôpital's rule:

$$= \frac{1}{n^2+\alpha^2} \lim_{z \to n} \frac{\pi}{\pi \sec^2(\pi z)}$$

$$= \frac{\sec^2(\pi n)}{n^2+\alpha^2}$$

$$= \frac{1}{n^2+\alpha^2}$$

Now we have all residues of f calculated, and we know that

$$\sum_j \operatorname*{Res}_{z=z_j} f(z) = -\frac{\pi\coth(\pi\alpha)}{2\alpha} - \frac{\pi\coth(\pi\alpha)}{2\alpha} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2+\alpha^2}$$

$$= -\frac{\pi\coth(\pi\alpha)}{\alpha} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2+\alpha^2}$$

In order for γ, a circle, to contain <u>all</u> of f's poles, we must give it a very large radius R, letting $R \to \infty$. Thus, we have

$$\lim_{R\to\infty} \oint_{|z|=R} f(z)\,dz = -\frac{\pi\coth(\pi\alpha)}{\alpha} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2+\alpha^2}$$

By parametrizing, we see that the integral is equal to

$$\int_0^{2\pi} \frac{\pi\cot(\pi R e^{i\theta})}{R^2 e^{2i\theta} + \alpha^2} \cdot iRe^{i\theta}\, d\theta$$

as $R \to \infty$, the integrand approaches zero, and so does the integral. Thus, we have

$$-\frac{\pi\coth(\pi\alpha)}{\alpha} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2+\alpha^2} = 0$$

or

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+\alpha^2} = \frac{\pi\coth(\pi\alpha)}{\alpha}$$

or

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n^2+\alpha^2} = \frac{\pi\coth(\pi\alpha)}{2\alpha} + \frac{1}{2\alpha^2}}$$

This result is one that I have not seen derived before without using the Residue Theorem. With this amazing result, I conclude the post. Keep in mind that my presentation of the theorem was definitely <u>non-rigorous</u>, and that it was meant to be an intuitive explanation. I encourage the reader to try and use the same strategy as I used here to evaluate other infinite series like the following:

$$\sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n)} \qquad \sum_{n=0}^{\infty} \frac{1}{n^4+1} \qquad \sum_{n=-\infty}^{\infty} \frac{1}{2n^3+1}$$