CONTINUOUS FUNCTION WITH DIVERGENT FOURIER SERIES

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This project tackles the question "when does the Fourier Series of a function $f : \mathbb{T} \to \mathbb{R}$ converge to the correct value at a point?" In other words, what conditions on f are needed to guarantee that

$$\sum_{|n|<N} \hat{f}(n)e^{inx} = \int_{-\pi}^{\pi} f(y)D_N(x-y)dy$$

converges to $f(x)$ as N approaches infinity? In class, we proved easily that $f \in C^2(\mathbb{T})$ is sufficient, and with a bit more difficulty that $f \in C^1(\mathbb{T})$ is also sufficient. However, surprisingly, continuity alone does not guarantee pointwise convergence of a function's Fourier series.

We start by constructing a counterexample that takes advantage of the unboundedness of the L^1 norm of the Dirichlet Kernels D_N and diverges at a point despite being continuous. However, we manage to salvage a slightly more conservative criterion for convergence: a continuous function must be "badly behaved" in other ways in order to have a divergent Fourier series at a point. Namely, it cannot be locally Hölder continuous and it must oscillate infinitely many times in a neighborhood of that point. Informally, the Fourier series of a continuous function is guaranteed to converge pointwise so long as that function is "not too steep or infinitely wiggly".

Of the requirements determining a "good kernel", the Dirichlet Kernels D_N fail to satisfy the property of having a bounded L^1 norm. That is,

$$\int_{-\pi}^{\pi} |D_N(x)|dx \to \infty$$

as $N \to \infty$. It also has an unbounded L^2 norm:

$$\int_{-\pi}^{\pi} D_N(x)^2 dx \to \infty$$

In order for a function $f : \mathbb{T} \to \mathbb{R}$ to have a divergent Fourier series at $x = 0$, the following integral must diverge as $N \to \infty$:

$$S_N f(0) = \int_{-\pi}^{\pi} f(x)D_N(x)dx$$

Intuitively, one way to design a function f which forces the above integral to diverge is to construct it in such a way that it "oscillates similarly" to D_N for infinitely many values of N. If f has the same sign as D_N on some interval, then $f(x)D_N(x)$ is positive on that interval, behaving similarly to $|D_N(x)|$ or $D_N(x)^2$, whose integrals grow unboundedly. So if f oscillates similarly to D_N for infinitely many values of N and on sufficiently large intervals,

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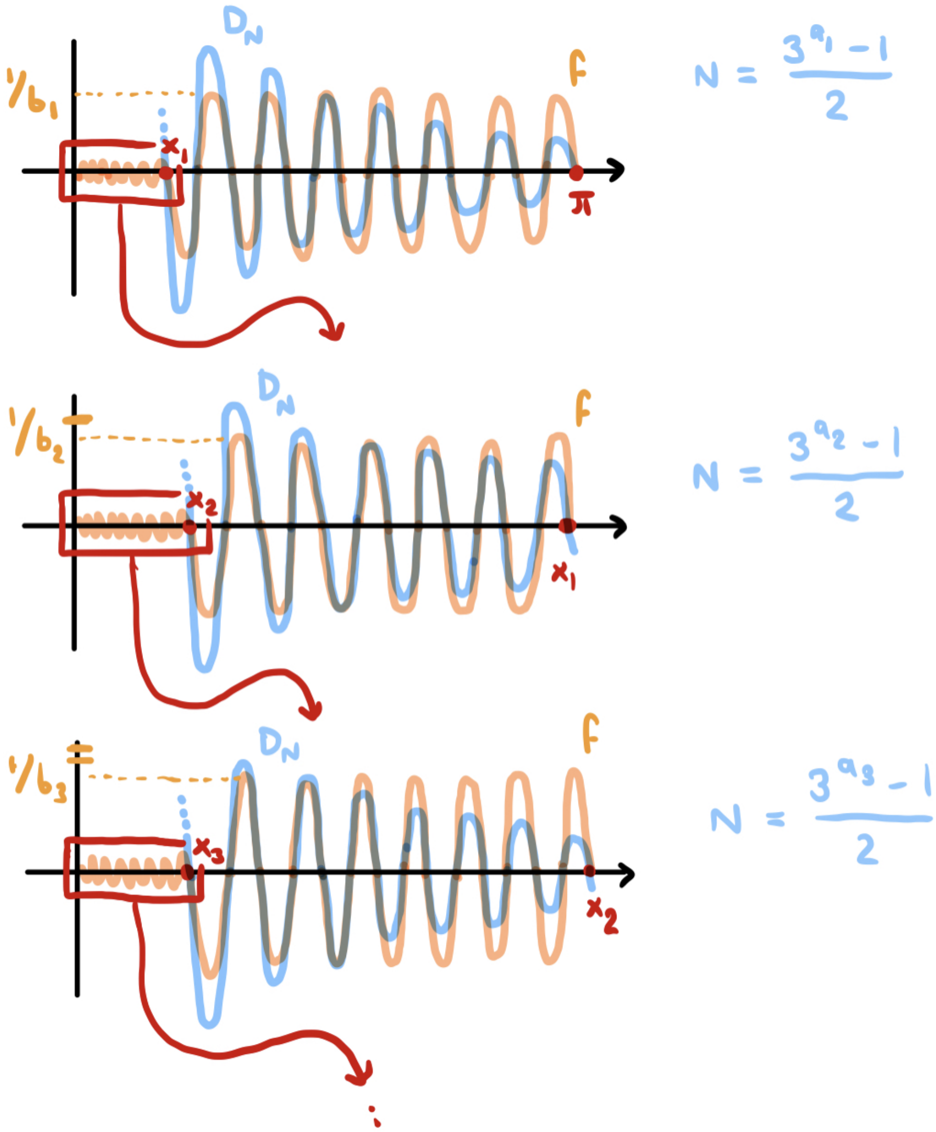
the integrals of $f(x)D_N(x)$ over those intervals will "blow up", while the rest of the domain of integration becomes negligible due to concentration of mass.



By dividing the domain $[0, \pi)$ into regimes $[x_{n+1}, x_n)$ of exponentially decreasing size, and tailoring the values of f in each regime to a different Dirichlet Kernel D_N, we can construct a function with precisely this behavior.

Theorem 1.1. *Let* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{N}$ *be an increasing sequence with* $a_1 = 1$ *and the property that* $a_n/a_{n-1} = \omega(b_n) \to \infty$ *as* $n \to \infty$ *for some increasing sequence* $\{b_n\}_{n=1}^{\infty} \subset \mathbb{N}$*, and define* $x_n = 2\pi/3^{a_n} \forall n \in \mathbb{N}$*. Finally, define the function* $f : \mathbb{T} \to \mathbb{R}$ *as follows:*

$$f(x) = \begin{cases} 0 & x \in [x_1, 2\pi) \cup \{0\} \\ \frac{1}{b_n} \sin(3^{a_n} x/2) & x \in [x_n, x_{n-1}), n > 1 \end{cases}$$

Then f *is continuous on* $\mathbb{T}$ *and its partial Fourier sums* $S_N f(x)$ *are unbounded at* $x = 0$.

Proof. First we show that f is continuous on $\mathbb{T}$. Because f is sinusoidal and hence continuous on each of the intervals (x_n, x_{n-1}), we need only check continuity at each of the points x_n and at $x = 0$.

On $[x_n, x_{n-1})$ we have $f(x) = \frac{1}{b_n}\sin(3^{a_n}x/2)$, and on $[x_{n+1}, x_n)$ we have $f(x) = \frac{1}{b_{n+1}}\sin(3^{a_{n+1}x/2})$. Thus, we have

$$\begin{aligned}\lim_{x\to x_n^+} f(x) &= \lim_{x\to x_n^+} \frac{1}{b_n}\sin(3^{a_n}x/2)\\ &= \frac{1}{b_n}\sin(3^{a_n}x_n/2)\\ &= \frac{1}{b_n}\sin(\pi)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to x_n^-} f(x) &= \lim_{x\to x_n^-} \frac{1}{b_{n+1}}\sin(3^{a_{n+1}}x/2)\\ &= \frac{1}{b_{n+1}}\sin(3^{a_{n+1}}x_n/2)\\ &= \frac{1}{b_{n+1}}\sin(3^{a_{n+1}-a_n}\pi)\\ &= 0\end{aligned}$$

Since these limits agree, we have continuity at each of the points $x = x_n$. Finally, we must check continuity at $x = 0$. In the regime $[x_n, x_{n-1})$, the magnitude of f is bounded above by $1/b_n$, which is $o(1)$, and since $x_n \to 0$ as $n \to \infty$, we have that $f(x)$ approaches zero as $x \to 0$ from above. On the other hand, $f(x) = 0$ for x in $[2\pi/3, 2\pi)$, so $f(x)$ approaches zero as $x \to 2\pi$ or $x \to 0$ from below. Hence f is also continuous at $x = 0$, and we have that it is continuous on all of $\mathbb{T}$.

Now we show that the partial Fourier sums of f diverge at $x = 0$. The partial Fourier sum $S_N f(0)$ is given by

$$S_N f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(x)D_N(x)dx$$

or, since $f = 0$ when $x > 2\pi/3$,

$$S_N f(0) = \frac{1}{2\pi}\int_0^{2\pi/3} f(x)D_N(x)dx$$

Let $n \in \mathbb{N}$, and choose N such that $2N + 1 = 3^{a_n}$. Then we have that

$$D_N(x) = \frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin(\frac{x}{2})} = \frac{\sin(3^{a_n}x/2)}{\sin(x/2)}$$

so that

$$S_N f(0) = \frac{1}{2\pi} \int_0^{2\pi/3} f(x) \frac{\sin(3^{a_n} x/2)}{\sin(x/2)}$$

We may split this integral into 3 by partitioning the interval $[0, 2\pi)$ into the 3 regimes $[0, x_n)$, $[x_n, x_{n-1})$ and $[x_{n-1}, 2\pi/3)$:

$$\begin{aligned} S_N f(0) &= \frac{1}{2\pi} \int_0^{2\pi/3} f(x) \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} \\ &= \frac{1}{2\pi} \left(\int_0^{x_n} + \int_{x_n}^{x_{n-1}} + \int_{x_{n-1}}^{2\pi/3} \right) f(x) \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} \\ &= \frac{I_1 + I_2 + I_3}{2\pi} \end{aligned}$$

Let us consider these 3 integrals separately, starting with I_1. Because the sine wave is bounded by 1 in magnitude everywhere on $\mathbb{T}$, and $1/b_n$ is $o(1)$, we have that $f(x)$ o(1) as $x \to 0$, or as $n \to \infty$ for $x \in [0, x_n)$. Thus,

$$|I_1| \leq \int_0^{x_n} o(1) \cdot |D_N(x)| dx$$

Further, we have that $D_N(x)$ is bounded in magnitude by $2N + 1$, meaning that

$$\begin{aligned} |I_1| &\leq \int_0^{x_n} o(1) \cdot 3^{a_n} dx \\ &= o(1) \cdot 3^{a_n} x_n \\ &= o(1) \cdot 2\pi \\ &= o(1) \end{aligned}$$

Hence, we have that $|I_1|$ is $o(1)$.

Next consider I_2. We may bound it as follows:

$$\begin{aligned} I_2 &= \int_{x_n}^{x_{n-1}} f(x) \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} dx \\ &= \frac{1}{b_n} \int_{x_n}^{x_{n-1}} \sin(3^{a_n} x/2) \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} dx \\ &= \frac{1}{b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \int_{2\pi k/3^{a_n}}^{2\pi(k+1)/3^{a_n}} \sin(3^{a_n} x/2) \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} dx \\ &= \frac{1}{b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \frac{1}{2 \cdot 3^{a_n}} \int_{\pi k}^{\pi(k+1)} \sin(x) \frac{\sin(x)}{\sin(x/3^{a_n})} dx \end{aligned}$$

$$
\begin{aligned}
&\text{(i)} && = \frac{1}{b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \frac{1}{2 \cdot 3^{a_n}} \int_0^\pi \frac{\sin^2(x)}{\sin\left(\frac{x+\pi k}{3^{a_n}}\right)} dx \\
&\text{(ii)} && \geq \frac{1}{b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \frac{1}{2 \cdot 3^{a_n}} \int_0^\pi \frac{\sin^2(x)}{\left(\frac{x+\pi k}{3^{a_n}}\right)} dx \\
& && = \frac{1}{2b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \int_0^\pi \frac{\sin^2(x)}{x + \pi k} dx \\
& && \geq \frac{1}{2b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \int_0^\pi \frac{\sin^2(x)}{\pi(k+1)} dx \\
& && = \left(\frac{1}{2\pi} \int_0^\pi \sin^2(x) dx \right) \cdot \frac{1}{b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \frac{1}{k+1} \\
& && = \Omega\left(\frac{1}{b_n} \sum_{k=1}^{3^{a_n - a_{n-1}} - 1} \frac{1}{k+1} \right) \\
&\text{(iii)} && = \Omega\left(\frac{\log(3^{a_n - a_{n-1}} - 1)}{b_n} \right) \\
& && = \Omega\left(\frac{a_n - a_{n-1}}{b_n} \right) \\
& && = \omega(a_{n-1})
\end{aligned}
$$

where (i) uses the fact that $\sin^2(x + \pi k) = \sin^2(x)\ \forall x \in \mathbb{R}$ for $k \in \mathbb{Z}$, (ii) uses the fact that $\sin x \leq x \forall x > 0$, and (iii) uses the fact that the mth partial sum of the harmonic series is $\sim \log(m)$. So we have that $I_2 = \omega(a_{n-1})$, and I_2 is also positive.

Finally, we consider I_3. Since $f(x)$ is bounded in magnitude by 1, we have that

$$
\begin{aligned}
|I_3| &\leq \int_{x_{n-1}}^{2\pi/3} \left| \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} \right| dx \\
&= \sum_{k=3^{a_n - a_{n-1}}}^{3^{a_n} - 1} \int_{2\pi k/3^{a_n}}^{2\pi(k+1)/3^{a_n}} \left| \frac{\sin(3^{a_n} x/2)}{\sin(x/2)} \right| dx \\
&= \sum_{k=3^{a_n - a_{n-1}}}^{3^{a_n} - 1} \frac{1}{2 \cdot 3^{a_n}} \int_{\pi k}^{\pi(k+1)} \left| \frac{\sin(x)}{\sin(x/3^{a_n})} \right| dx \\
&= \sum_{k=3^{a_n - a_{n-1}}}^{3^{a_n} - 1} \frac{1}{2 \cdot 3^{a_n}} \int_0^\pi \left| \frac{\sin(x)}{\sin(\frac{x+\pi k}{3^{a_n}})} \right| dx \\
&\leq \sum_{k=3^{a_n - a_{n-1}}}^{3^{a_n} - 1} \frac{1}{2 \cdot 3^{a_n}} \int_0^\pi \left| \frac{\sin(x)}{\frac{2}{\pi} \cdot \frac{x+\pi k}{3^{a_n}}} \right| dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n}-1} \frac{\pi}{4}\int_0^\pi \left|\frac{\sin(x)}{x+\pi k}\right| dx \\
&\le \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n}-1} \frac{\pi}{4}\int_0^\pi \left|\frac{\sin(x)}{\pi k}\right| dx \\
&= \frac{1}{2}\sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n}-1} \frac{1}{k} \\
&= \mathcal{O}\Big(\log(3^{a_n}-1) - \log(3^{a_n-a_{n-1}})\Big) \\
&= \mathcal{O}(a_{n-1})
\end{aligned}
$$

Thus, we have that $|I_3|$ is $\mathcal{O}(a_{n-1})$. Combining these three bounds, we have that

$$S_N f(0) = o(1) + \omega(a_{n-1}) + \mathcal{O}(a_{n-1})$$

or

$$S_N f(0) = \omega(a_{n-1})$$

which tends to infinity as $n \to \infty$. Thus, we have that $S_N f(0)$ approaches infinity as $n \to \infty$, and we have found an unbounded subsequence of the sequence of partial Fourier sums at $x = 0$, implying that they do not converge. □

Corollary 1.2. *There exists a continuous function on $\mathbb{T}$ whose Fourier series diverges at $x = 0$.*

Proof. Define the function f as it is defined in Theorem 1.1, letting $a_n = 2^{n^2}$. We have that

$$
\begin{aligned}
\frac{a_n - a_{n-1}}{n} &= \frac{2^{n^2} - 2^{(n-1)^2}}{n} \\
&= 2^{(n-1)^2} \cdot \frac{2^{2n-1} - 1}{n} \\
&= \Omega\Big(2^{(n-1)^2}\Big) \\
&= \Omega(a_{n-1})
\end{aligned}
$$

meaning that a_n satisfies the growth conditions necessary for Theorem 1.1, and hence the function f has a divergent Fourier series at $x = 0$. □

Hence, we have constructed a continuous function with a divergent Fourier series at $x = 0$. Note, however, that $S_N f(0)$ does not approach ∞ as $N \to \infty$. For if this were true, it would necessarily be true that the Fejer means $F_N f(0)$ would also approach infinity - but it is a theorem that the Fejer sums of continuous functions always converge. So, since we designed

f in such a way that the partial Fourier sums $S_N f$ have a subsequence tending to ∞, we may conclude that the partial Fourier sums diverge, but not to ∞.

Although it is tragic that continuity does not guarantee a pointwise convergent Fourier series, the counterexample that we managed to construct, despite being continuous, is pathological in several ways. For one, it displays oscillatory behavior with greater and greater frequencies approaching $x = 0$. Additionally, although it equals zero at $x = 0$, its decay is incredibly slow - that is, it is not Hölder continuous for any Hölder exponent (this can be verified as an exercise). Must all counterexamples necessarily be pathological in these ways?

In the following section we will affirm this conjecture, but not before proving several lemmas to help with the proof.

Lemma 1.3. *Let $f : [0, 1] \to \mathbb{R}$ be a monotone increasing function continuous at $x = 0$ with $f(0) = 0$, and let $\{a_n\}_{n=0}^{\infty}$ be an $o(1)$ sequence. Then*

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left(f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right) a_k = 0$$

Proof. We will show that the quantity

$$S_n = \sum_{k=0}^{n-1} \left(f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right) a_k$$

satisfies $|S_n| \le \epsilon$ for all $n \ge N$ for some $N \in \mathbb{N}$, for any $\epsilon > 0$. We have that

$$|S_n| \le \sum_{k=0}^{n-1} \left|f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right| |a_k|$$

by the triangle inequality. Now, given $\epsilon > 0$, define the natural numbers N as follows:

(1) Let $N_1 \in \mathbb{N}$ be such that $|a_n| < \epsilon/2f(1)$ for all $n \ge N_1$. (Such N_1 exists because a_n tends to zero as $n \to \infty$.)
(2) Let $N_2 \in \mathbb{N}$ be such that $f(N_1/n) \le \epsilon/2N_1 a$ for all $n \ge N_2$, where $a = \sup\{|a_n|\}_{n=0}^{N_1}$. (Such N_2 exists because $f(0) = 0$ and f is continuous at 0.)

Now let $N = \max(N_1, N_2) + 1$, so that N satisfies both of the above properties. Then, for $n \ge N$, we may split up the summation as follows:

$$|S_n| \le \sum_{k=0}^{N_1-1} \left|f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right| |a_k| + \sum_{k=N_1}^{n-1} \left|f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right| |a_k|$$

In the first sum, we have that $\left|f(\frac{k+1}{n}) - f(\frac{k}{n})\right|$ is less than or equal to $\left|f(\frac{k+1}{n})\right|$ since f is monotone increasing and nonnegative, which in turn is less than $\epsilon/2N_1 a$. Further, the quantity $|a_k|/a$ is less than or equal to 1 for all $k \le N_1$, by the definition of N_1. Thus, we have that each term in the first sum is less than or equal to $\epsilon/2N_1$, and the first sum is less than or equal to $\epsilon/2$:

$$|S_n| \le \frac{\epsilon}{2} + \sum_{k=N_1}^{n-1} \left|f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right| |a_k|$$

Now, in the second sum, $|a_k| \leq \epsilon/2f(1)$. Additionally, since f is monotone increasing and nonnegative, we have that the quantity $f(\frac{k+1}{n}) - f(\frac{k}{n})$ is nonnegative and therefore equal to its absolute value:

$$|S_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2f(1)} \sum_{k=N_1}^{n-1} \left(f(\tfrac{k+1}{n}) - f(\tfrac{k}{n})\right)$$

This is now a telescoping sum, whose value is equal to $f(1) - f(\frac{N_1+1}{n})$, which is less than or equal to $f(1)$, meaning that the entire summation term is less than or equal to $f(1)\epsilon/2f(1) = \epsilon/2$, implying that

$$|S_n| \leq \epsilon$$

as desired. Hence, since there exists an N for any $\epsilon > 0$ such that $|S_n| \leq \epsilon$ for all $n \geq N$, we have that S_n tends to 0 as $n \to \infty$. □

Lemma 1.4. *Let D_N be the Nth Dirichlet Kernel. Then the area of its nth "hump" in $[0, \pi)$ is given by*

$$\int_{x_n}^{x_{n+1}} D_N(x)dx = (-1)^n \Theta\left(\frac{1}{n}\right)$$

where $x_n = \frac{\pi n}{N+1/2}$, for $k = 0, ..., n$, and the constant used in Θ is independent of N.

Proof. We have the following:

$$\begin{aligned}
\int_{x_n}^{x_{n+1}} D_N(x)dx &= \int_{x_n}^{x_{n+1}} \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}dx \\
&= \Theta\left(\int_{x_n}^{x_{n+1}} \frac{\sin(\frac{2N+1}{2}x)}{x}dx\right) \qquad \text{(i)} \\
&= \Theta\left(\int_{\pi n}^{\pi(n+1)} \frac{\sin(x)}{x}dx\right) \\
&= \Theta\left((-1)^n \int_0^{\pi} \frac{\sin(x)}{x+\pi n}dx\right) \qquad \text{(ii)} \\
&= \Theta\left(\frac{(-1)^n}{n} \int_0^{\pi} \frac{\sin(x)}{\pi + \frac{x}{n}}dx\right) \\
&= (-1)^n \Theta\left(\frac{1}{n}\right)
\end{aligned}$$

where (i) uses the fact that $\frac{2}{\pi}x < \sin x < x$ for all $x \in (0, \pi/2)$, and (ii) uses the fact that $\sin(x + \pi n) = (-1)^n \sin x$. □

Lemma 1.5. *Let D_N and x_n be as defined in Lemma 1.4. Then the difference in the areas in two consecutive "humps" of D_n can be bounded by*

$$\int_{x_n}^{x_{n+1}} D_N(x)dx + \int_{x_{n+1}}^{x_{n+2}} D_N(x)dx = (-1)^n \Theta\left(\frac{1}{n^2}\right)$$

where x_n is defined as in Lemma 1.4.

Proof. We may derive the bounds as follows:

$$\begin{aligned}
\int_{x_n}^{x_{n+1}} D_N(x)dx + \int_{x_{n+1}}^{x_{n+2}} D_N(x)dx &= \int_{x_n}^{x_{n+1}} \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}dx + \int_{x_{n+1}}^{x_{n+2}} \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}dx \\
&= \frac{1}{N+1/2}\int_{\pi n}^{\pi(n+1)} \frac{\sin(x)}{\sin(\frac{x}{2N+1})}dx + \int_{\pi(n+1)}^{\pi(n+2)} \frac{\sin(x)}{\sin(\frac{x}{2N+1})}dx \\
&= \frac{1}{N+1/2}\int_0^{\pi} \frac{\sin(x+\pi n)}{\sin(\frac{x+\pi n}{2N+1})}dx + \int_0^{\pi} \frac{\sin(x+\pi(n+1))}{\sin(\frac{x+\pi(n+1)}{2N+1})}dx \\
&= \frac{(-1)^n}{N+1/2}\int_0^{\pi} \sin(x)\left(\frac{1}{\sin(\frac{x+\pi n}{2N+1})} - \frac{1}{\sin(\frac{x+\pi(n+1)}{2N+1})}\right)dx \\
&= \frac{(-1)^n}{N+1/2}\int_0^{\pi} \sin(x)\cdot\frac{\sin(\frac{x+\pi(n+1)}{2N+1}) - \sin(\frac{x+\pi n}{2N+1})}{\sin(\frac{x+\pi n}{2N+1})\sin(\frac{x+\pi(n+1)}{2N+1})}dx \\
\text{(i)} \quad &= \frac{(-1)^n}{N+1/2}\int_0^{\pi} \sin(x)\cdot\frac{\cos(\frac{x+\pi n}{2N+1})\cdot\frac{\pi}{2N+1} + \mathcal{O}(\frac{1}{(2N+1)^2})}{\sin(\frac{x+\pi n}{2N+1})\sin(\frac{x+\pi(n+1)}{2N+1})}dx \\
\text{(ii)} \quad &= \frac{(-1)^n}{N+1/2}\Theta\left(\int_0^{\pi} \sin(x)\cdot\frac{(\frac{\pi}{2} - \frac{x+\pi n}{2N+1})\cdot\frac{\pi}{2N+1} + \mathcal{O}(\frac{1}{(2N+1)^2})}{\frac{x+\pi n}{2N+1}\cdot\frac{x+\pi(n+1)}{2N+1}}dx\right) \\
\text{(iii)} \quad &= \frac{(-1)^n}{N+1/2}\Theta\left(\int_0^{\pi} \sin(x)\cdot\Theta\left(\frac{N}{n^2}+\frac{1}{n}\right)dx\right) \\
&= \frac{(-1)^n}{N+1/2}\Theta\left(\frac{N}{n^2}+\frac{1}{n}\right) \\
&= (-1)^n\Theta\left(\frac{1}{n^2}\right)
\end{aligned}$$

Where (i) follows from the Talor Series of the sine and the fact that it has bounded derivatives on $[0,\pi]$; (ii) follows from the inequality $\frac{2}{\pi}(\frac{\pi}{2}-x) \leq \cos(x) \leq \frac{\pi}{2}-x$; and (iii) follows by letting $n > 1$ and using the fact that $x \in [0,\pi]$ in the integrand. □

Theorem 1.6. *If $f : \mathbb{T} \to \mathbb{R}$ is monotone in $[0,\epsilon]$ and $[-\epsilon, 0]$ for some $\epsilon > 0$ and continuous at $x = 0$ with $f(0) = 0$, then the partial Fourier sums $S_N f(0) \to 0$ as $N \to \infty$.*

Proof. Let $f : \mathbb{T} \to \mathbb{R}$ be a function that is monotone increasing on $[0,\epsilon]$ for some $\epsilon > 0$. Define the function $f^* : \mathbb{T} \to \mathbb{R}$ as follows:

$$f^*(x) = \begin{cases} f(x) & x \in [0,\epsilon] \\ f(\epsilon) & x \in (\epsilon, 2\pi) \end{cases}$$

so that f^* is monotone on $[0, 2\pi)$. We also have that

$$\int_0^\pi f(x)D_N(x)dx = \int_0^\pi f^*(x)D_N(x)dx + o(1)$$

or, equivalently, $S_N f(0) = S_N f^*(0) + o(1)$ as $N \to \infty$, which follows from accumulation of mass of the Dirichlet Kernel about $x = 0$. Thus, the convergence of $S_N f(0)$ is equivalent to that of $S_N f^*(0)$ and their limits are equal to each other if they converge, so we may work with $S_N f^*(0)$ instead. Now, we have that

$$\begin{aligned} S_N f^*(0) &= \int_{-\pi}^{\pi} f^*(x)D_N(x)dx \\ &= \int_{-\pi}^{0} f^*(x)D_N(x)dx + \int_0^\pi f^*(x)D_N(x)dx \\ &= I_- + I_+ \end{aligned}$$

First consider the integral I_+, which we may bound as follows:

$$\begin{aligned} |I_+| &= \left| \int_0^\pi f^*(x)D_N(x)dx \right| \\ &= o(1) + \left| \int_0^{\frac{\pi N}{N+1/2}} f^*(x)D_N(x)dx \right| \\ &= o(1) + \left| \sum_{k=0}^{N-1} \int_{\frac{\pi k}{N+1/2}}^{\frac{\pi(k+1)}{N+1/2}} f^*(x)D_N(x)dx \right| \\ &= o(1) + \left| \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x)D_N(x)dx + \int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x)D_N(x)dx \right| \\ &= o(1) + \left| \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left(f^*(x) - f^*(x + \tfrac{\pi}{N+1/2})\right) D_N(x)dx + \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x + \tfrac{\pi}{N+1/2}) D_N(x)dx \right. \\ &\qquad \left. + \int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x)D_N(x)dx \right| \\ &\le o(1) + \left| \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left(f^*(x) - f^*(x + \tfrac{\pi}{N+1/2})\right) D_N(x)dx \right| \\ &\qquad + \left| \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x + \tfrac{\pi}{N+1/2}) D_N(x)dx + \int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x)D_N(x)dx \right| \\ &= o(1) + |J_1| + |J_2| \end{aligned}$$

where

$$J_1 = \sum_{k=0}^{\lfloor N/2\rfloor-1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big(f^*(x) - f^*(x+\tfrac{\pi}{N+1/2})\Big) D_N(x)dx$$

and

$$J_2 = \sum_{k=0}^{\lfloor N/2\rfloor-1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x+\tfrac{\pi}{N+1/2}) D_N(x)dx + \int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x)D_N(x)dx$$

We now show that $|J_1|$ and $|J_2|$ both approach 0 as $N \to \infty$. Firstly, for $|J_1|$, we have by the triangle inequality that

$$|J_1| \leq \sum_{k=0}^{\lfloor N/2\rfloor-1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big|f^*(x) - f^*(x+\tfrac{\pi}{N+1/2})\Big|\Big|D_N(x)\Big|dx$$

Now, because f^* is monotonic, the quantity $|f(x)-f(y)|$ is bounded above by $|f(a)-f(b)|$ for all x, y in an interval $[a, b]$. This means that

$$\begin{aligned} |J_1| &\leq \sum_{k=0}^{\lfloor N/2\rfloor-1} \Big|f^*(\tfrac{2k\pi}{N+1/2}) - f^*(\tfrac{(2k+2)\pi}{N+1/2})\Big| \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big|D_N(x)\Big|dx \\ &= \sum_{k=0}^{\lfloor N/2\rfloor-1} \Big|f^*(\tfrac{2k\pi}{N+1/2}) - f^*(\tfrac{(2k+2)\pi}{N+1/2})\Big| \Theta\Big(\frac{1}{k}\Big) \end{aligned}$$

Finally, since the constants used in Θ in the above sum do not depend on N, and since f^* is monotone increasing and continuous at $x = 0$ with $f^*(0) = 0$, it follows easily from Lemma 1.1 that the above sum tends to zero as $N \to \infty$.

Next we consider $|J_2|$. Using a substitution, the integrals can be combined to form

$$J_2 = \sum_{k=0}^{\lfloor N/2\rfloor-1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x+\tfrac{\pi}{N+1/2})\big(D_N(x) + D_N(x+\tfrac{\pi}{N+1/2})\big)dx$$

Now, it can be seen that $D_N(x)$ is positive and greater in magnitude than $D_N(x+\frac{\pi}{N+1/2})$ for each x in the interval of integration, ultimately following from the fact that $k < N/2$ and that $\sin(x)$ is a strictly increasing function on $[0, \pi/2]$. This implies that the integrand of the above integral is always positive. Further, since f^* is monotone increasing and therefore always nonnegative, we have that

$$J_2 = |J_2| \leq \sum_{k=0}^{\lfloor N/2\rfloor-1} f^*\Big(\tfrac{(2k+2)\pi}{N+1/2}\Big) \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \big(D_N(x) + D_N(x+\tfrac{\pi}{N+1/2})\big)dx$$

Now, since this integral is equivalent to the one bounded in Lemma 1.5, we have that

$$|J_2| \leq \sum_{k=0}^{\lfloor N/2\rfloor-1} f^*\Big(\tfrac{(2k+2)\pi}{N+1/2}\Big)\Theta\Big(\frac{1}{k^2}\Big)$$

Again, we may apply Lemma 1.1 to argue that this sum tends to zero as $N \to \infty$, since the constants of the Θ term do not depend on N.

Thus, we have shown that both $|J_1|$ and $|J_2|$ approach zero as $N \to \infty$, implying that $|I_+| \to 0$ as well, as desired. Now, if f were monotone decreasing function on $[0, \epsilon]$ instead of monotone increasing, we could use the same argument to show that the partial Fourier sums of $-f$, a monotone increasing function, tend to zero. Since $S_N(-f) = -S_N f$, this would imply that $S_N f \to 0$ as well. Hence, the result holds for both monotone increasing and monotone decreasing functions on $[0, \epsilon]$ for any $\epsilon > 0$. Similarly, to show that $|I_-| \to 0$, we may note that the integral I_- is equivalent to the integral I_+ with $f(x)$ replaced by $f(-x)$. Since monotonicity of $f(x)$ on $[0, \epsilon]$ is equivalent to monotonicity of $f(-x)$ on $[-\epsilon, 0]$, we may use the same argument again to show that $|I_-|$ tends to zero as $N \to \infty$.

Finally, this allows us to conclude that for any function $f : \mathbb{T} \to \mathbb{R}$ that is monotone on $[0, \epsilon]$ and $[-\epsilon, 0]$ for some $\epsilon > 0$, and that further vanishes and is continuous at $x = 0$, its partial Fourier sums $S_N f$ must converge to 0 as $N \to \infty$. □

Corollary 1.7. *If $f : \mathbb{T} \to \mathbb{R}$ is continuous at $a \in \mathbb{T}$ and monotone in the intervals $[a, a+\epsilon]$ and $[a-\epsilon, a]$ for some $\epsilon > 0$, then the partial Fourier sums $S_N f(a) \to f(a)$ as $N \to \infty$.*

Proof. The function $g(x) = f(x+a) - f(a)$ satisfies the hypothesis of Theorem 1.4, and $S_N g(0) = S_N f(a) - f(a)$, implying that $S_N f(a) - f(a)$ tends to zero, and $S_N f(a) \to f(a)$ as $N \to \infty$. □

Theorem 1.8. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function with $f(0) = 0$ and satisfying the following "logarithmic continuity condition": for all x in some neighborhood of 0,*

$$|f(x+h) - f(x)| = o\Big(\frac{1}{\log h}\Big)$$

as $h \to 0$. Then the partial Fourier sums $S_N f(0) \to 0$ as $N \to \infty$.

Proof. Suppose that the function $f : \mathbb{T} \to \mathbb{R}$ satisfies $f(0) = 0$ as well as the following continuity condition for x in the neighborhood $(-\epsilon, \epsilon)$ for some $\epsilon > 0$:

$$|f(x+h) - f(x)| = o\Big(\frac{1}{\log h}\Big)$$

as $h \to 0$. In particular, by letting $x = 0$, it follows that we may choose an even smaller neighborhood $(-\epsilon', \epsilon') \subset (-\epsilon, \epsilon)$ with $0 < \epsilon' < \epsilon$ such that f satisfies

$$|f(h)| \le -\frac{C}{\log h}$$

for some constant $C \ge 0$ and all $h \in (-\epsilon', \epsilon)$. Now, let us define the function f^* as follows:

$$f^*(x) = \begin{cases} f(-\epsilon') & x \in (-\pi, -\epsilon'] \\ f(x) & x \in (-\epsilon', \epsilon') \\ f(\epsilon') & x \in [\epsilon', \pi] \end{cases}$$

Since f satisfies the logarithmic continuity condition in the problem statement for $x \in (-\epsilon', \epsilon')$, and f^* is equal to f on this interval and constant everywhere else, it follows trivially that f^* satisfies the continuity condition for all $x \in \mathbb{T} \backslash \{\pi\}$.

Following the same vein as in the proof of Theorem 1.4, we may say by concentration of mass of the Dirichlet Kernel that

$$\int_0^{\pi} f(x)D_N(x)dx = \int_0^{\pi} f^*(x)D_N(x)dx + o(1)$$

or that $S_N f(0) = S_N f^*(0) + o(1)$, so it suffices to show that $S_N f^*(0) \to 0$ as $N \to \infty$.

If we proceed as in Theorem 1.4 and define $|I_+|$, $|I_-|$, $|J_1|$, and $|J_2|$ identically, we obtain the bounds

$$|J_1| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left| f^*(x) - f^*(x + \tfrac{\pi}{N+1/2}) \right| \left| D_N(x) \right| dx$$

$$|J_2| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left| f^*(x + \tfrac{\pi}{N+1/2}) \right| \left(D_N(x) + D_N(x + \tfrac{\pi}{N+1/2}) \right) dx$$

Let us start by considering the bound on $|J_1|$. From the logarithmic continuity condition, we have that

$$\left| f^*(x) - f^*(x + \tfrac{\pi}{N+1/2}) \right| = o\Big(\frac{1}{\log N}\Big)$$

for $x \in (-\pi, \pi)$. Hence, we have

$$\begin{aligned} |J_1| &\leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} o\Big(\frac{1}{\log N}\Big) \left| D_N(x) \right| dx \\ &= o\Big(\frac{1}{\log N}\Big) \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left| D_N(x) \right| dx \\ &= o\Big(\frac{1}{\log N}\Big) \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \Theta\Big(\frac{1}{k}\Big) \\ &= o\Big(\frac{1}{\log N}\Big) \Theta\big(\log N\big) \\ &= o(1) \end{aligned}$$

and so $|J_1| \to 0$, where we have used the bounds on the "hump areas" of the Dirichlet Kernel derived in Lemma 1.4.

Next we consider the bounds on $|J_2|$. First of all, leveraging concentration of mass of the Dirichlet Kernel again, we need only consider the first $\lfloor N\epsilon'/2\pi \rfloor - 1$ terms of this sum of integrals:

$$|J_2| \leq o(1) + \sum_{k=0}^{\lfloor N\epsilon'/2\pi \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left| f^*(x + \tfrac{\pi}{N+1/2}) \right| \left(D_N(x) + D_N(x + \tfrac{\pi}{N+1/2}) \right) dx$$

This restriction guarantees that $x \in (-\epsilon', \epsilon)$ for all x in the interval of integration of each integral in the sum. Hence, we may apply the logarithmic bounds on f^*:

$$\begin{aligned}
|J_2| &\le o(1) + \sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left|f^*(x+\tfrac{\pi}{N+1/2})\right|\big(D_N(x) + D_N(x+\tfrac{\pi}{N+1/2})\big)dx \\
&\le o(1) - \sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \frac{C}{\ln(\frac{(2k+2)\pi}{N+1/2})}\big(D_N(x) + D_N(x+\tfrac{\pi}{N+1/2})\big)dx \\
&\le o(1) - \sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor - 1} \frac{C}{\ln(x+\frac{\pi}{N+1/2})} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \big(D_N(x) + D_N(x+\tfrac{\pi}{N+1/2})\big)dx \\
&= o(1) - \sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor - 1} \frac{C}{\ln(\frac{(2k+2)\pi}{N+1/2})}\Theta\Big(\frac{1}{k^2}\Big)
\end{aligned}$$

which, again, is $o(1)$ as $N \to \infty$, meaning that $|J_2| \to 0$. Hence, we have that $|I_+| \to 0$ as $N \to \infty$ as well, and we may similarly argue that $|I_-| \to 0$ as $N \to \infty$ by considering the function $f(-x)$, which satisfies the same logarithmic continuity condition. Hence, we have the desired result: that $S_N f(0) \to 0$ as $N \to \infty$. □

Corollary 1.9. *If $f : \mathbb{T} \to \mathbb{R}$ satisfies the logarithmic continuity condition*

$$|f(x+h) - f(x)| = o\Big(\frac{1}{\log h}\Big)$$

as $h \to 0$ for all x in some neighborhood of $x = a$, then $S_N f(a) \to f(a)$ as $N \to \infty$.

Proof. The function $g(x) = f(x+a) - f(a)$ satisfies the hypothesis of Theorem 1.6, and $S_N g(0) = S_N f(a) - f(a)$, implying that $S_N f(a) - f(a) \to 0$ as $N \to \infty$, and hence $S_N f(a) \to f(a)$ as desired. □

Corollary 1.10. *If $f : \mathbb{T} \to \mathbb{R}$ is Lipschitz or Hölder continuous at a point, then its partial Fourier sums converge to the correct value at that point.*