CONTINUOUS FUNCTION WITH DIVERGENT FOURIER SERIES

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1. Introduction

This project tackles the question "when does the Fourier Series of a function $f : \mathbb{T} \to \mathbb{R}$ converge to the correct value at a point?" In other words, what conditions on f are needed to guarantee that

$$\sum_{|n|<N} \hat{f}(n)e^{inx} = \int_{-\pi}^{\pi} f(y)D_N(x-y)dy$$

converges to $f(x)$ as N approaches infinity? In class, we proved easily that $f \in C^2(\mathbb{T})$ is sufficient, and cited (without proof) the known result that $f \in C^1(\mathbb{T})$ is also sufficient for uniform convergence. However, surprisingly, continuity alone does not guarantee pointwise convergence of a function's Fourier series.

We start by constructing a counterexample that takes advantage of the unboundedness of the L^1 norm of the Dirichlet Kernels D_N and diverges at a point despite being continuous. However, we manage to salvage a slightly more conservative criterion for convergence: a continuous function must be "badly behaved" in other ways in order to have a divergent Fourier series at a point. Namely, it cannot be locally Hölder continuous and it must oscillate infinitely many times in a neighborhood of that point. Informally, the Fourier series of a continuous function is guaranteed to converge pointwise so long as that function is neither "too steep" nor "infinitely wiggly".

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2. Preliminaries

In this section, we'll go over some notation and basic results that allow us to construct our counterexample later.

Definition 2.1. *The following notation is used to describe asymptotic behavior of functions:*

- ***Big-O****: we write that $f(x) = \mathcal{O}(g(x))$ as $x \to a$ (or $x \to \infty$) if $|f(x)| < C \cdot g(x)$ for all x sufficiently close to a (or sufficiently large) for some constant $C > 0$.*
- ***Little-O****: we write that $f(x) = o(g(x))$ as $x \to a$ (or $x \to \infty$) if $|f(x)| < C \cdot g(x)$ for all x sufficiently close to a (or sufficiently large) for any constant $C > 0$.*

Date: December 17, 2021.

- ***Big-Omega**: we write that $f(x) = \Omega(g(x))$ as $x \to a$ (or $x \to \infty$) if $|f(x)| > C \cdot g(x)$ for all x sufficiently close to a (or sufficiently large) for some constant $C > 0$.*
- ***Little-Omega**: we write that $f(x) = \omega(g(x))$ as $x \to a$ (or $x \to \infty$) if $|f(x)| > C \cdot g(x)$ for all x sufficiently close to a (or sufficiently large) for any constant $C > 0$.*
- ***Big-Theta**: we write that $f(x) = \Theta(g(x))$ as $x \to a$ (or $x \to \infty$) if $C_1 \cdot g(x) < f(x) < C_2 \cdot g(x)$ for all x sufficiently close to a (or sufficiently large) for some constants $C_1, C_2 > 0$.*

Intuitively, each of these notations can be interpreted as follows:

- $f(x) = \mathcal{O}(g(x))$ means that f does not asymptotically dominate g
- $f(x) = o(g(x))$ means that f is asymptotically dominated by g
- $f(x) = \Omega(g(x))$ means that g does not asymptotially dominate f
- $f(x) = \omega(g(x))$ means that f asymptotically dominates g
- $f(x) = \Theta(g(x))$ means that f and g do not asymptotically dominate each other, or that they are of the "same growth order"

We will use each of these notations for convenience in analyzing the asymptotic behavior of functions. For instance, we will use them to show that a certain subsequence of the partial Fourier sums of our counterexample construction tends towards infinity and therefore has a divergent Fourier series.

We also make use of the Dirichlet Kernel and its relationship to the partial Fourier sums of functions $f : \mathbb{T} \to \mathbb{R}$, both of which are described in greater detail in [2].

Definition 2.2. *The **Dirichlet Kernels** $D_N : \mathbb{T} \to \mathbb{R}$ are defined for $N \in \mathbb{N}$ as*

$$D_N(x) = \sum_{|n| \le N} e^{inx}$$

Proposition 2.3. *The Dirichlet Kernels are given by the formula*

$$D_N(x) = \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}$$

and are bounded in magnitude by $|D_N(x)| \le 2N + 1$ for all $x \in \mathbb{T}$.

Proposition 2.4. *The partial Fourier sums of any integrable function $f : \mathbb{T} \to \mathbb{R}$ are given by its convolutions with the Dirichlet Kernels:*

$$S_N f(x) = \sum_{|n| \le N} \hat{f}(n) e^{inx} = \int_{-\pi}^{\pi} f(y) D_N(x - y) dy$$

Finally, here are a few elementary facts which we will not prove, but which are used in proofs to come:

Proposition 2.5. *For all $x \in \mathbb{R}$ with $x > 0$, we have that $\sin(x) < x$.*

Proposition 2.6. *For all $x \in \mathbb{R}$ with $x \in [0, \pi/2]$, we have that $\sin(x) \geq \frac{2}{\pi}x$, and equality holds precisely at $x = 0$ and $x = \pi/2$.*

Proposition 2.7. *The partial sums of the harmonic series diverge logarithmically. That is,*

$$\sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$$

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3. Intuition

Of the requirements determining a "good kernel", the Dirichlet Kernels D_N fail to satisfy the property of having a bounded L^1 norm. That is,

$$\int_{-\pi}^{\pi} |D_N(x)| dx \to \infty$$

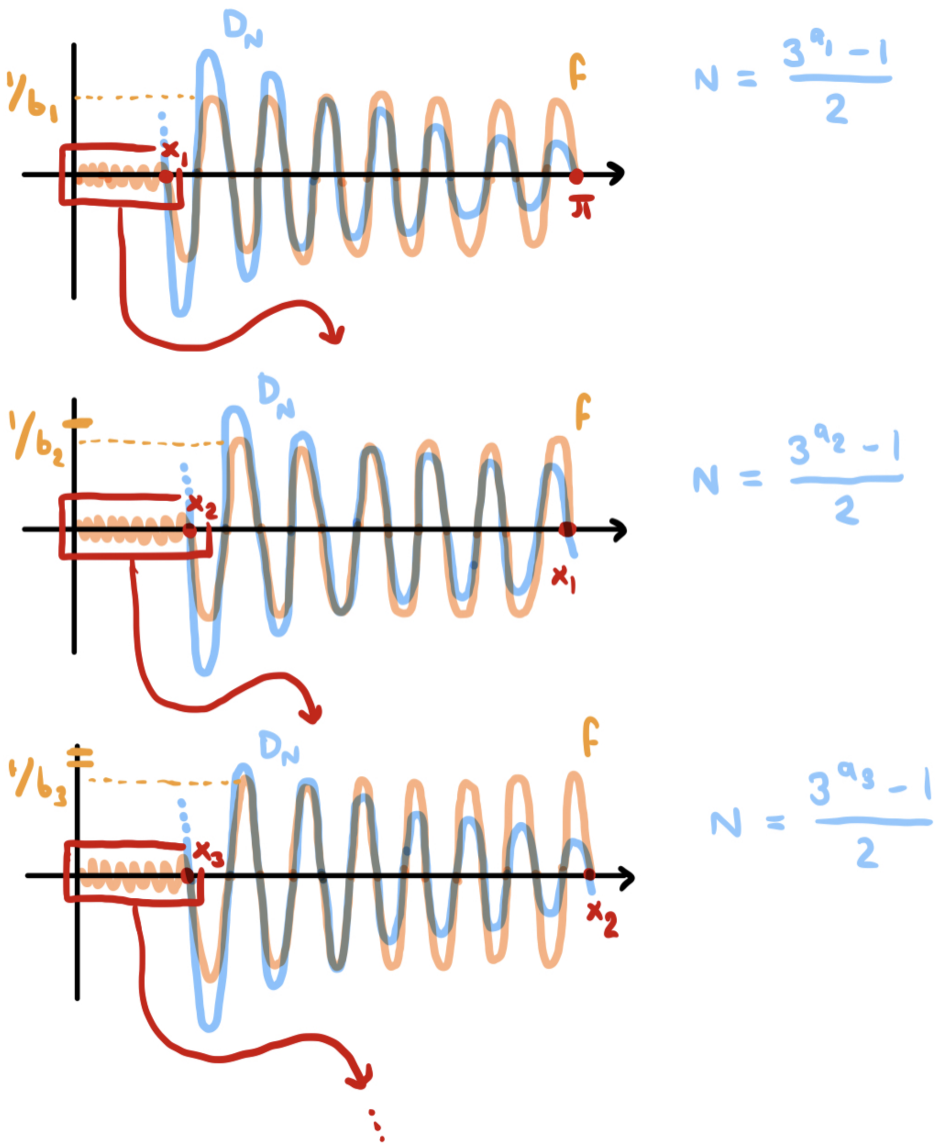
as $N \to \infty$. It also has an unbounded L^2 norm:

$$\int_{-\pi}^{\pi} D_N(x)^2 dx \to \infty$$

In order for a function $f : \mathbb{T} \to \mathbb{R}$ to have a divergent Fourier series at $x = 0$, the following integral must diverge as $N \to \infty$:

$$S_N f(0) = \int_{-\pi}^{\pi} f(x) D_N(x) dx$$

Intuitively, one way to design a function f which forces the above integral to diverge is to construct it in such a way that it "oscillates similarly" to D_N for infinitely many values of N. If f has the same sign as D_N on some interval, then $f(x)D_N(x)$ is positive on that interval, behaving similarly to $|D_N(x)|$ or $D_N(x)^2$, whose integrals grow unboundedly. So if f oscillates similarly to D_N for infinitely many values of N and on sufficiently large intervals, the integrals of $f(x)D_N(x)$ over those intervals will "blow up", while the rest of the domain of integration becomes negligible due to concentration of mass.



By dividing the domain $[0, \pi)$ into regimes $[x_{n+1}, x_n)$ of exponentially decreasing size, and tailoring the values of f in each regime to a different Dirichlet Kernel D_N, we can construct a function with precisely this behavior.

4. Counterexample construction

Theorem 4.1. *Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{N}$ be a strictly increasing sequence with $a_1 = 1$ and the property that $a_n/a_{n-1} = \omega(b_n) \to \infty$ as $n \to \infty$ for some increasing sequence $\{b_n\}_{n=1}^{\infty} \subset \mathbb{N}$, and define $x_n = 2\pi/3^{a_n} \forall n \in \mathbb{N}$. Finally, define the function $f : \mathbb{T} \to \mathbb{R}$ as follows:*

$$f(x) = \begin{cases} 0 & x \in [x_1, 2\pi) \cup \{0\} \\ \frac{1}{b_n} \sin(3^{a_n} x/2) & x \in [x_n, x_{n-1}), n > 1 \end{cases}$$

Then f is continuous on $\mathbb{T}$ and its partial Fourier sums $S_N f(x)$ are unbounded at $x = 0$.

Proof. First we show that f is continuous on $\mathbb{T}$. Because f is sinusoidal and hence continuous on each of the intervals (x_n, x_{n-1}), we need only check continuity at each of the points x_n and at $x = 0$.

On $[x_n, x_{n-1})$ we have $f(x) = \frac{1}{b_n}\sin(3^{a_n}x/2)$, and on $[x_{n+1}, x_n)$ we have $f(x) = \frac{1}{b_{n+1}}\sin(3^{a_{n+1}x/2})$. Thus, since $x_n = 2\pi/3^{a_n}$, we have that

$$\begin{aligned}
\lim_{x\to x_n^+} f(x) &= \lim_{x\to x_n^+}\frac{1}{b_n}\sin(3^{a_n}x/2)\\
&= \frac{1}{b_n}\sin(3^{a_n}x_n/2)\\
&= \frac{1}{b_n}\sin(\pi)\\
&= 0
\end{aligned}$$

and further, since (a_n) is a strictly increasing sequence of integers, $a_{n+1} - a_n$ is a positive natural number and $3^{a_{n+1}-a_n}$ is a natural number, meaning that

$$\begin{aligned}
\lim_{x\to x_n^-} f(x) &= \lim_{x\to x_n^-}\frac{1}{b_{n+1}}\sin(3^{a_{n+1}}x/2)\\
&= \frac{1}{b_{n+1}}\sin(3^{a_{n+1}}x_n/2)\\
&= \frac{1}{b_{n+1}}\sin(3^{a_{n+1}-a_n}\pi)\\
&= 0
\end{aligned}$$

Since these limits agree, we have continuity at each of the points $x = x_n$. Finally, we must check continuity at $x = 0$. In the regime $[x_n, x_{n-1})$, the magnitude of f is bounded above by $1/b_n$, which is $o(1)$ as $n \to \infty$ since (b_n) is a strictly increasing sequence of natural numbers. Further, since $x_n \to 0$ as $n \to \infty$, we have that $f(x)$ approaches zero as $x \to 0$ from above. On the other hand, $f(x) = 0$ for x in $[2\pi/3, 2\pi)$, so $f(x)$ approaches zero as $x \to 2\pi$ or $x \to 0$ from below. Hence f is also continuous at $x = 0$, and we have that it is continuous on all of $\mathbb{T}$.

Now we show that the partial Fourier sums of f diverge at $x = 0$. The partial Fourier sum $S_N f(0)$ is given by

$$S_N f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(x)D_N(x)dx$$

or, since $f = 0$ when $x > 2\pi/3$,

$$S_N f(0) = \frac{1}{2\pi}\int_0^{2\pi/3} f(x) D_N(x) dx$$

To show divergence, it is sufficient to show that this expression is unbounded in N, and that it tends to ∞ for some infinite sequence of N-valued. Let us consider a the subsequence of N-values (N_n) defined by $2N_n + 1 = 3^{a_n}$ (which is valid since 3^{a_n} is odd). Then we have that

$$D_N(x) = \frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin(\frac{x}{2})} = \frac{\sin(3^{a_n}x/2)}{\sin(x/2)}$$

so that

$$S_N f(0) = \frac{1}{2\pi}\int_0^{2\pi/3} f(x)\frac{\sin(3^{a_n}x/2)}{\sin(x/2)}dx$$

where $N = N_n$ varies with n (although we write N in place of N_n for convenience).

We may split this integral into 3 pieces by partitioning the interval $[0, 2\pi)$ into the 3 regimes $[0, x_n)$, $[x_n, x_{n-1})$ and $[x_{n-1}, 2\pi/3)$:

$$\begin{aligned}
S_N f(0) &= \frac{1}{2\pi}\int_0^{2\pi/3} f(x)\frac{\sin(3^{a_n}x/2)}{\sin(x/2)}dx \\
&= \frac{1}{2\pi}\left(\int_0^{x_n} + \int_{x_n}^{x_{n-1}} + \int_{x_{n-1}}^{2\pi/3}\right) f(x)\frac{\sin(3^{a_n}x/2)}{\sin(x/2)}dx \\
&= \frac{I_1 + I_2 + I_3}{2\pi}
\end{aligned}$$

Let us consider these 3 integrals separately, starting with I_1. Because the sine wave is bounded by 1 in magnitude everywhere on $\mathbb{T}$, and $1/b_n$ is $o(1)$, we have that $f(x)$ is $o(1)$ as $x \to 0$, or as $n \to \infty$ for $x \in [0, x_n)$. Thus,

$$|I_1| \le \int_0^{x_n} o(1) \cdot |D_N(x)| dx$$

Further, we have that $D_N(x)$ is bounded in magnitude by $2N + 1$, meaning that

$$\begin{aligned}
|I_1| &\le \int_0^{x_n} o(1) \cdot 3^{a_n} dx \\
&= o(1) \cdot 3^{a_n} x_n \\
&= o(1) \cdot 2\pi \\
&= o(1)
\end{aligned}$$

Hence, we have that $|I_1|$ is $o(1)$.

Next consider I_2. We may bound it as follows, using the fact that $f(x) = \frac{1}{b_n}\sin(3^{a_n}x/2)$ on $[x_n, x_{n-1})$. By making the substitution $x \mapsto 2x/3^{a_n}$, we have

$$\begin{aligned}
I_2 &= \int_{x_n}^{x_{n-1}} f(x)\frac{\sin(3^{a_n}x/2)}{\sin(x/2)}dx \\
&= \frac{1}{b_n}\int_{x_n}^{x_{n-1}} \sin(3^{a_n}x/2)\frac{\sin(3^{a_n}x/2)}{\sin(x/2)}dx \\
&= \frac{1}{b_n}\int_{3^{a_n}x_n/2}^{3^{a_n}x_{n-1}/2} \sin(x)\frac{\sin(x)}{\sin(x/3^{a_n})}\cdot\frac{2}{3^{a_n}}dx \\
&= \frac{1}{b_n}\int_{\pi}^{3^{a_n-a_{n-1}}\pi} \sin(x)\frac{\sin(x)}{\sin(x/3^{a_n})}\cdot\frac{2}{3^{a_n}}dx
\end{aligned}$$

again using the fact that $x_n = 2\pi/3^{a_n}$. Now, by partitioning the interval of this integral into pieces of length π and using the fact that $\sin(x) < x$ for $x > 0$, we have that

$$\begin{aligned}
I_2 &= \frac{1}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}}-1}\frac{2}{3^{a_n}}\int_{\pi k}^{\pi(k+1)} \sin(x)\frac{\sin(x)}{\sin(x/3^{a_n})}dx \\
&= \frac{1}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}}-1}\frac{2}{3^{a_n}}\int_0^{\pi} \frac{\sin^2(x)}{\sin\left(\frac{x+\pi k}{3^{a_n}}\right)}dx \\
&\geq \frac{1}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}}-1}\frac{2}{3^{a_n}}\int_0^{\pi} \frac{\sin^2(x)}{\left(\frac{x+\pi k}{3^{a_n}}\right)}dx \\
&= \frac{2}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}}-1}\int_0^{\pi} \frac{\sin^2(x)}{x+\pi k}dx \\
&\geq \frac{2}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}}-1}\int_0^{\pi} \frac{\sin^2(x)}{\pi(k+1)}dx \\
&= \left(\frac{2}{\pi}\int_0^{\pi}\sin^2(x)dx\right)\cdot\frac{1}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}}-1}\frac{1}{k+1}
\end{aligned}$$

Since $\frac{2}{\pi}\int_0^{\pi}\sin^2(x)dx$ is just a (positive) constant factor, we may simplify this expression by converting it into a big-Ω asymptotic formula. Since I_2 is greater than or equal to a constant factor times the sum divided by b_n, we have that

$$I_2 = \Omega\left(\frac{1}{b_n}\sum_{k=1}^{3^{a_n-a_{n-1}-1}}\frac{1}{k+1}\right)$$

Further, since the partial sums of the harmonic series diverge logarithmically, we may simplify this as follows:

$$I_2 = \Omega\left(\frac{\log(3^{a_n-a_{n-1}}-1)}{b_n}\right)$$

Now, since a_n/a_{n-1} tends to infinity, it follows that $a_n - a_{n-1}$ tends to infinity, meaning that $\log(3^{a_n-a_{n-1}}-1)$ behaves asymptotically like $\log(3)(a_n - a_{n-1})$, which is a constant factor times $a_n - a_{n-1}$. Hence,

$$I_2 = \Omega\left(\frac{a_n - a_{n-1}}{b_n}\right)$$

Again, since a_n/a_{n-1} is $\omega(b_n)$ by hypothesis, it follows that $\frac{a_n-a_{n-1}}{b_n}$ is $\omega(a_{n-1})$, so we have that $I_2 = \omega(a_{n-1})$, and I_2 is also positive.

Finally, we consider I_3. We will use a technique very similar to the one we used to bound I_2, with the primary difference being that we seek to bound I_3 above rather than below (to the end of showing that it does not grow fast enough to "interfere with" the growth of I_2). By using the fact that f is bounded in magnitude by 1 and making the same substitution $x \mapsto 2x/3^{a_n}$ as before, we have that

$$\begin{aligned}
|I_3| &\le \int_{x_{n-1}}^{2\pi/3}\left|\frac{\sin(3^{a_n}x/2)}{\sin(x/2)}\right|dx \\
&\le \int_{3^{a_n}x_{n-1}/2}^{3^{a_n-1}\pi}\left|\frac{\sin(x)}{\sin(x/3^{a_n})}\right|\cdot\frac{2}{3^{a_n}}dx \\
&= \int_{3^{a_n-a_{n-1}}\pi}^{3^{a_n-1}\pi}\left|\frac{\sin(x)}{\sin(x/3^{a_n})}\right|\cdot\frac{2}{3^{a_n}}dx
\end{aligned}$$

Now, partitioning this integral into intervals of length π as before, we obtain the following sum:

$$|I_3| \le \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n-1}-1}\frac{1}{2\cdot 3^{a_n}}\int_{\pi k}^{\pi(k+1)}\left|\frac{\sin(x)}{\sin(x/3^{a_n})}\right|dx$$

Now, using the fact that $\sin(x) > \frac{2}{\pi}x$ on $(0,\pi/2)$, we have that

$$
\begin{aligned}
|I_3| &< \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n-1}-1} \frac{1}{2 \cdot 3^{a_n}} \int_{\pi k}^{\pi(k+1)} \left| \frac{\sin(x)}{\frac{2x}{\pi 3^{a_n}}} \right| dx \\
&= \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n-1}-1} \pi \int_{\pi k}^{\pi(k+1)} \left| \frac{\sin(x)}{x} \right| dx
\end{aligned}
$$

Now, on the interval $[\pi k, \pi(k+1)]$, x is at least πk, meaning that $1/x$ is at most $1/\pi k$, so that we have

$$|I_3| < \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n-1}-1} \pi \int_{\pi k}^{\pi(k+1)} \left| \frac{\sin(x)}{\pi k} \right| dx$$

Finally, pulling the constant factor out of the integrals and using the fact that $\int_{\pi k}^{\pi(k+1)} |\sin x| dx = 2$ for any k, we have

$$|I_3| < 2 \sum_{k=3^{a_n-a_{n-1}}}^{3^{a_n-1}-1} \frac{1}{k}$$

This is a difference of two partial sums of the harmonic series! Using the logarithmic divergence of the harmonic series once more, we conclude that

$$|I_3| = \mathcal{O}\Big(\log(3^{a_n} - 1) - \log(3^{a_n-a_{n-1}}) \Big)$$

Which, by similar reasoning as before, implies that $|I_3|$ is $\mathcal{O}(a_{n-1})$.

Combining these three bounds, we have that

$$S_N f(0) = o(1) + \omega(a_{n-1}) + \mathcal{O}(a_{n-1})$$

for the specified subsequence N_n of N-values, which means that

$$S_N f(0) = \omega(a_{n-1})$$

which tends to infinity as $n \to \infty$. Thus, we have that $S_N f(0)$ approaches infinity as $n \to \infty$ for $N = N_n$, and we have found an unbounded subsequence of the sequence of partial Fourier sums at $x = 0$, implying that they do not converge. □

Corollary 4.2. *There exists a continuous function on $\mathbb{T}$ whose Fourier series diverges at $x = 0$.*

Proof. Define the function f as it is defined in Theorem 1.1, letting $a_n = 2^{n^2}$. We have that

$$\frac{a_n}{a_{n-1}} = \frac{2^{n^2}}{2^{(n-1)^2}}$$

$$= 2^{2n-1}$$
$$= \omega(n)$$
$$= \omega(b_n)$$

meaning that a_n satisfies the growth conditions necessary for Theorem 1.1 (with a choice of $b_n = n$), and hence the function f described previously has a divergent Fourier series at $x = 0$. $\square$

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5. Discussion

Hence, we have constructed a continuous function with a divergent Fourier series at $x = 0$. Note, however, that $S_N f(0)$ does not approach ∞ as $N \to \infty$. For if this were true, it would necessarily be true that the Fejer means $\sigma_N f(0)$ would also approach infinity - but it is a theorem that the Fejer sums of continuous functions always converge. So, since we designed f in such a way that the partial Fourier sums $S_N f$ have a subsequence tending to ∞, we may conclude that the partial Fourier sums diverge, but not to ∞.

The above construction, it turns out, is not a new one. DuBois-Reymond described a similar piecewise construction in his 1873 paper *Über die fourierschen Reihen* (although his construction is not accompanied by a detailed proof). He also begins with the idea of a function which behaves like a sinusoid whose frequency grows boundlessly and amplitude decays to zero as it approaches the origin:

> ...setzen wir $f(\alpha) = \rho(\alpha) \sin \psi(\alpha)$ und nehmen zunächst an, dass $\psi(\alpha)$ für $\alpha = 0$ ohne Maxima unendlich und $\rho(\alpha)$ ebenso Null wird. Dann wird auch die Dichtigkeit der Maxima von $f(\alpha) = \rho(\alpha) \sin \psi(\alpha)$ für $\alpha = 0$ ohne Maxima unendlich.

He then notices that the convolutions of f with the Nth Dirichlet Kernel can be made sufficiently large in magnitude on some family of subintervals of $[0, 2\pi)$ that the magnitude of the convolution on remainder of the interval of integration cannot cancel it out:

> ...dies führt zu der Einsicht, dass, wenn der Limes von
> $$\int_0^a d\alpha f(\alpha) \frac{\sin h\alpha}{\sin \alpha}$$
> unendlich werden soll, man dies nur dem Theile:
> $$\int_{x'}^{x''}$$
> wird verdanken können, weil hier keine negativen Theile die positiven aufheben oder umgekehrt.

He also chooses to define f piecewise on a sequence of intervals shrinking towards the origin, with the frequency ψ constant on each interval:

> ...ferner seien $k_1, k_2, ...$ Grössen, welche die Bedingung
> $$k_1 > k_2 > ... > k_\infty = \infty$$
> erfüllen. Da die Dichtigkeit der Maxima von $\sin kx$ für jeden Werth von k constant ist, so wird eine Funktion $f(x)$, welche in den Intervallen $\Lambda_1, \Lambda_2, ...$ resp. die Werthe $\sin k_1 x, \sin k_2 x, ...$ erhält, in jedem dieser Intervalle constante Dichtigkeit ihrer Maxima haben, und diese Dichtigkeit wird von Intervall zu Intervall springen, bis zu schliesslich unendlichen Werthen.

In this way, his construction is essentially the same as ours, aside from the fact that his choice of partitioning points and frequencies is slightly different:

> ...nunmehr findet den allerdings ein Unendlichwerden des Integrals:
> $$\int_{x_p}^{x_{p-1}} d\alpha \rho(\alpha) \sin \psi(\alpha) \frac{\sin h\alpha}{\sin \alpha}$$
> statt, z.B. wenn:
> $$x_p = \frac{a}{\prod_0^{p-1}(2^q + 1)}, \ k_p = \frac{1}{x_{p-1}x_p}$$
> angenommen wird.

After finishing my write-up of the above proof, it felt incredibly surreal to dig up DuBois-Reymond's paper, written several decades ago and in German, and discover that the two of us had arrived at essentially the exact same construction! [1] Perhaps this coincidence suggests that the piecewise sinusoidal construction outlined above is a somewhat "naturally occurring" counterexample.

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6. Alternate conditions for convergence

Although it is tragic that continuity does not guarantee a pointwise convergent Fourier series, the counterexample that we managed to construct, despite being continuous, is pathological in several ways. For one, it displays oscillatory behavior with greater and greater frequencies approaching $x = 0$. Additionally, although it equals zero at $x = 0$, its decay is incredibly slow - that is, it is not Hölder continuous for any Hölder exponent (this can be verified as an exercise). Must all counterexamples necessarily be pathological in these ways?

In the following section we will affirm this conjecture, but not before proving several lemmas to help with the proof.

Lemma 6.1. *Let $f : [0, 1] \to \mathbb{R}$ be a monotone increasing function continuous at $x = 0$ with $f(0) = 0$, and let $\{a_n\}_{n=0}^{\infty}$ be an $o(1)$ sequence. Then*

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\big(f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big)a_k=0$$

Proof. We will show that the quantity

$$S_n=\sum_{k=0}^{n-1}\big(f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big)a_k$$

satisfies $|S_n|\le\epsilon$ for all $n\ge N$ for some $N\in\mathbb{N}$, for any $\epsilon>0$. We have that

$$|S_n|\le\sum_{k=0}^{n-1}\big|f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big||a_k|$$

by the triangle inequality. Now, given $\epsilon>0$, define the natural numbers N_1,N_2 as follows:

(1) Let $N_1\in\mathbb{N}$ be such that $|a_n|<\epsilon/2f(1)$ for all $n\ge N_1$. (Such N_1 exists because a_n tends to zero as $n\to\infty$.)
(2) Let $N_2\in\mathbb{N}$ be such that $f(N_1/n)\le\epsilon/2N_1a$ for all $n\ge N_2$, where $a=\sup\{|a_n|\}_{n=0}^{N_1}$. (Such N_2 exists because $f(0)=0$ and f is continuous at 0.)

Now let $N=\max(N_1,N_2)+1$, so that N satisfies both of the above properties. Then, for $n\ge N$, we may split up the summation as follows:

$$|S_n|\le\sum_{k=0}^{N_1-1}\big|f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big||a_k|+\sum_{k=N_1}^{n-1}\big|f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big||a_k|$$

In the first sum, we have that $\big|f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big|$ is less than or equal to $\big|f(\tfrac{k+1}{n})\big|$ since f is monotone increasing and nonnegative, which in turn is less than $\epsilon/2N_1a$. Further, the quantity $|a_k|/a$ is less than or equal to 1 for all $k\le N_1$, by the definition of N_1. Thus, we have that each term in the first sum is less than or equal to $\epsilon/2N_1$, and the first sum is less than or equal to $\epsilon/2$:

$$|S_n|\le\frac{\epsilon}{2}+\sum_{k=N_1}^{n-1}\big|f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big||a_k|$$

Now, in the second sum, $|a_k|\le\epsilon/2f(1)$. Additionally, since f is monotone increasing and nonnegative, we have that the quantity $f(\frac{k+1}{n})-f(\frac{k}{n})$ is nonnegative and therefore equal to its absolute value:

$$|S_n|\le\frac{\epsilon}{2}+\frac{\epsilon}{2f(1)}\sum_{k=N_1}^{n-1}\big(f(\tfrac{k+1}{n})-f(\tfrac{k}{n})\big)$$

This is now a telescoping sum, whose value is equal to $f(1) - f(\frac{N_1+1}{n})$, which is less than or equal to $f(1)$, meaning that the entire summation term is less than or equal to $f(1)\epsilon/2f(1) = \epsilon/2$, implying that

$$|S_n| \leq \epsilon$$

as desired. Hence, since there exists an N for any $\epsilon > 0$ such that $|S_n| \leq \epsilon$ for all $n \geq N$, we have that S_n tends to 0 as $n \to \infty$. □

Lemma 6.2. *Let $f : [0, 1] \to \mathbb{R}$ be a bounded function continuous at $x = 0$ with $f(0) = 0$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence for which $\sum_{n\geq 1} a_n$ converges absolutely. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\tfrac{k}{n})a_k = 0$$

Proof. Similarly to the previous lemma, we will show that

$$S_n = \sum_{k=1}^{n} f(\tfrac{k}{n})a_k$$

satisfies $|S_n| \leq \epsilon$ for all $n \geq N$ for some $N \in \mathbb{N}$, for any given $\epsilon > 0$. Applying the triangle inequality again, we have that

$$|S_n| \leq \sum_{k=1}^{n} \left|f(\tfrac{k}{n})\right||a_k|$$

Given arbitrary $\epsilon > 0$, define N_1, N_2 as follows:

(1) Let $N_1 \in \mathbb{N}$ be such that $\sum_{k\geq n} |a_k| < \epsilon/2M$ for all $n \geq N_1$, where M is an upper bound on the magnitude of f. (Such N_1 exists because $\sum a_k$ converges absolutely and because f is bounded.)
(2) Let $N_2 \in \mathbb{N}$ be such that $f(N_1/n) \leq \epsilon/2N_1a$ for all $n \geq N_2$, where $a = \sup\{|a_n|\}_{n=0}^{N_1}$. (Such N_2 exists because $f(0) = 0$ and f is continuous at 0.)

Now let $N = \max(N_1, N_2) + 1$, so that N satisfies both of the above properties. Then, for $n \geq N$, we may split up the summation as follows:

$$|S_n| \leq \sum_{k=1}^{N_1} \left|f(\tfrac{k}{n})\right||a_k| + \sum_{k=N_1+1}^{n} \left|f(\tfrac{k}{n})\right||a_k|$$

In the first sum, we have that $\left|f(\frac{k}{n})\right|$ is less than $\epsilon/2N_1a$ because $n \geq N \geq N_2$, and the quantity $|a_k|/a$ is less than or equal to 1 for all $k \leq N_1$ by the definition of N_1. This means that each term in the first sum is less than or equal to $\epsilon/2N_1$, and the first sum is less than or equal to $\epsilon/2$:

$$|S_n| \leq \frac{\epsilon}{2} + \sum_{k=N_1+1}^{n} \left|f(\tfrac{k}{n})\right||a_k|$$

Now, in the second sum, $\left|f(\frac{k}{n})\right|$ is bounded by M and $\sum_{k=N_1+1}^{n}|a_k|$ is at most $\epsilon/2M$ by the definition of N_1, so we have that

$$|S_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$$

as desired. Hence, since there exists an N for any $\epsilon > 0$ such that $|S_n| \leq \epsilon$ for all $n \geq N$, we have that S_n tends to 0 as $n \to \infty$. $\square$

Lemma 6.3. *Let D_N be the Nth Dirichlet Kernel. Then the area of its nth "hump" in $[0, \pi)$ is given by*

$$\int_{x_n}^{x_{n+1}} D_N(x)dx = (-1)^n \Theta\left(\frac{1}{n}\right)$$

where $x_n = \frac{\pi n}{N+1/2}$, for $k = 0, ..., n$, and the constant used in Θ is independent of N.

Proof. We have the following:

$$\begin{aligned}
\int_{x_n}^{x_{n+1}} D_N(x)dx &= \int_{x_n}^{x_{n+1}} \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}dx \\
\text{(i)} \qquad &= \Theta\left(\int_{x_n}^{x_{n+1}} \frac{\sin(\frac{2N+1}{2}x)}{x}dx\right) \\
&= \Theta\left(\int_{\pi n}^{\pi(n+1)} \frac{\sin(x)}{x}dx\right) \\
\text{(ii)} \qquad &= \Theta\left((-1)^n \int_0^{\pi} \frac{\sin(x)}{x+\pi n}dx\right) \\
&= \Theta\left(\frac{(-1)^n}{n} \int_0^{\pi} \frac{\sin(x)}{\pi + \frac{x}{n}}dx\right) \\
&= (-1)^n \Theta\left(\frac{1}{n}\right)
\end{aligned}$$

where (i) uses the fact that $\frac{2}{\pi}x < \sin x < x$ for all $x \in (0, \pi/2)$, and (ii) uses the fact that $\sin(x + \pi n) = (-1)^n \sin x$. $\square$

Lemma 6.4. *Let D_N and x_n be as defined in Lemma 1.4. Then the difference in the areas in two consecutive "humps" of D_n can be bounded by*

$$\int_{x_n}^{x_{n+1}} D_N(x)dx + \int_{x_{n+1}}^{x_{n+2}} D_N(x)dx = (-1)^n \Theta\left(\frac{1}{n^2}\right)$$

where x_n is defined as in Lemma 1.4 and the constants used in Θ are independent of N.

Proof. We may derive the bounds as follows:

$$\begin{aligned}
\int_{x_n}^{x_{n+1}} D_N(x)dx + \int_{x_{n+1}}^{x_{n+2}} D_N(x)dx &= \int_{x_n}^{x_{n+1}} \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}dx + \int_{x_{n+1}}^{x_{n+2}} \frac{\sin(\frac{2N+1}{2}x)}{\sin(\frac{1}{2}x)}dx \\
&= \frac{1}{N+1/2}\int_{\pi n}^{\pi(n+1)} \frac{\sin(x)}{\sin(\frac{x}{2N+1})}dx + \int_{\pi(n+1)}^{\pi(n+2)} \frac{\sin(x)}{\sin(\frac{x}{2N+1})}dx \\
&= \frac{1}{N+1/2}\int_0^{\pi} \frac{\sin(x+\pi n)}{\sin(\frac{x+\pi n}{2N+1})}dx + \int_0^{\pi} \frac{\sin(x+\pi(n+1))}{\sin(\frac{x+\pi(n+1)}{2N+1})}dx \\
&= \frac{(-1)^n}{N+1/2}\int_0^{\pi} \sin(x)\left(\frac{1}{\sin(\frac{x+\pi n}{2N+1})} - \frac{1}{\sin(\frac{x+\pi(n+1)}{2N+1})}\right)dx \\
&= \frac{(-1)^n}{N+1/2}\int_0^{\pi} \sin(x)\cdot\frac{\sin(\frac{x+\pi(n+1)}{2N+1}) - \sin(\frac{x+\pi n}{2N+1})}{\sin(\frac{x+\pi n}{2N+1})\sin(\frac{x+\pi(n+1)}{2N+1})}dx \\
\text{(i)} \qquad &= \frac{(-1)^n}{N+1/2}\int_0^{\pi} \sin(x)\cdot\frac{\cos(\frac{x+\pi n}{2N+1})\cdot\frac{\pi}{2N+1} + \mathcal{O}(\frac{1}{(2N+1)^2})}{\sin(\frac{x+\pi n}{2N+1})\sin(\frac{x+\pi(n+1)}{2N+1})}dx \\
\text{(ii)} \qquad &= \frac{(-1)^n}{N+1/2}\Theta\left(\int_0^{\pi} \sin(x)\cdot\frac{(\frac{\pi}{2} - \frac{x+\pi n}{2N+1})\cdot\frac{\pi}{2N+1} + \mathcal{O}(\frac{1}{(2N+1)^2})}{\frac{x+\pi n}{2N+1}\cdot\frac{x+\pi(n+1)}{2N+1}}dx\right) \\
\text{(iii)} \qquad &= \frac{(-1)^n}{N+1/2}\Theta\left(\int_0^{\pi} \sin(x)\cdot\Theta\left(\frac{N}{n^2} + \frac{1}{n}\right)dx\right) \\
&= \frac{(-1)^n}{N+1/2}\Theta\left(\frac{N}{n^2} + \frac{1}{n}\right) \\
&= (-1)^n\Theta\left(\frac{1}{n^2}\right)
\end{aligned}$$

Where (i) follows from the Talyor Series of the sine and the fact that it has bounded derivatives on $[0,\pi]$; (ii) follows from the inequality $\frac{2}{\pi}(\frac{\pi}{2}-x) \le \cos(x) \le \frac{\pi}{2} - x$; and (iii) follows by letting $n > 1$ and using the fact that $x \in [0,\pi]$ in the integrand. $\square$

Theorem 6.5. *If $f : \mathbb{T} \to \mathbb{R}$ is Riemann integrable on $\mathbb{T}$, monotone in $[0,\epsilon]$ and $[-\epsilon, 0]$ for some $\epsilon > 0$, and continuous at $x = 0$ with $f(0) = 0$, then the partial Fourier sums $S_N f(0) \to 0$ as $N \to \infty$.*

Proof. Let $f:\mathbb{T}\to\mathbb{R}$ be a function that is monotone increasing on $[0,\epsilon]$ for some $\epsilon>0$. Define the function $f^*:\mathbb{T}\to\mathbb{R}$ as follows:

$$f^*(x)=\begin{cases} f(x) & x\in[0,\epsilon] \\ f(\epsilon) & x\in(\epsilon,2\pi) \end{cases}$$

so that f^* is monotone on $[0,2\pi)$. We also have that

$$\int_0^\pi f(x)D_N(x)dx=\int_0^\pi f^*(x)D_N(x)dx+o(1)$$

or, equivalently, $S_Nf(0)=S_Nf^*(0)+o(1)$ as $N\to\infty$, which follows from accumulation of mass of the Dirichlet Kernel about $x=0$. Thus, the convergence of $S_Nf(0)$ is equivalent to that of $S_Nf^*(0)$ and their limits are equal to each other if they converge, so we may work with $S_Nf^*(0)$ instead. Now, we have that

$$\begin{aligned} S_Nf^*(0) &= \int_{-\pi}^{\pi} f^*(x)D_N(x)dx \\ &= \int_{-\pi}^{0} f^*(x)D_N(x)dx+\int_0^\pi f^*(x)D_N(x)dx \\ &= I_- + I_+ \end{aligned}$$

First consider the integral I_+, which we may bound as follows:

$$\begin{aligned} |I_+| &= \left|\int_0^\pi f^*(x)D_N(x)dx\right| \\ &= o(1)+\left|\int_0^{\frac{\pi N}{N+1/2}} f^*(x)D_N(x)dx\right| \\ &= o(1)+\left|\sum_{k=0}^{N-1}\int_{\frac{\pi k}{N+1/2}}^{\frac{\pi(k+1)}{N+1/2}} f^*(x)D_N(x)dx\right| \\ &= o(1)+\left|\sum_{k=0}^{\lfloor N/2\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x)D_N(x)dx+\int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x)D_N(x)dx\right| \\ &= o(1)+\left|\sum_{k=0}^{\lfloor N/2\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \left(f^*(x)-f^*(x+\tfrac{\pi}{N+1/2})\right)D_N(x)dx+\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x+\tfrac{\pi}{N+1/2})D_N(x)dx\right. \\ &\qquad \left. +\int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x)D_N(x)dx\right| \end{aligned}$$

$$\begin{aligned}
&\leq o(1) + \left| \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big(f^*(x) - f^*(x + \tfrac{\pi}{N+1/2}) \Big) D_N(x) dx \right| \\
&\quad + \left| \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x + \tfrac{\pi}{N+1/2}) D_N(x) dx + \int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x) D_N(x) dx \right| \\
&= o(1) + |J_1| + |J_2|
\end{aligned}$$

where

$$J_1 = \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big(f^*(x) - f^*(x + \tfrac{\pi}{N+1/2}) \Big) D_N(x) dx$$

and

$$J_2 = \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x + \tfrac{\pi}{N+1/2}) D_N(x) dx + \int_{\frac{(2k+1)\pi}{N+1/2}}^{\frac{(2k+2)\pi}{N+1/2}} f^*(x) D_N(x) dx$$

We now show that $|J_1|$ and $|J_2|$ both approach 0 as $N \to \infty$. Firstly, for $|J_1|$, we have by the triangle inequality that

$$|J_1| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big| f^*(x) - f^*(x + \tfrac{\pi}{N+1/2}) \Big| \Big| D_N(x) \Big| dx$$

Now, because f^* is monotonic, the quantity $|f(x) - f(y)|$ is bounded above by $|f(a) - f(b)|$ for all x, y in an interval $[a, b]$. This means that

$$\begin{aligned}
|J_1| &\leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \Big| f^*(\tfrac{2k\pi}{N+1/2}) - f^*(\tfrac{(2k+2)\pi}{N+1/2}) \Big| \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big| D_N(x) \Big| dx \\
&= \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \Big| f^*(\tfrac{2k\pi}{N+1/2}) - f^*(\tfrac{(2k+2)\pi}{N+1/2}) \Big| \Theta\Big(\frac{1}{k}\Big)
\end{aligned}$$

Finally, since the constants used in Θ in the above sum do not depend on N, and since f^* is monotone increasing and continuous at $x = 0$ with $f^*(0) = 0$, it follows easily from Lemma 1.3 that the above sum tends to zero as $N \to \infty$.

Next we consider $|J_2|$. Using a substitution, the integrals can be combined to form

$$J_2 = \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} f^*(x + \tfrac{\pi}{N+1/2}) \big(D_N(x) + D_N(x + \tfrac{\pi}{N+1/2}) \big) dx$$

Now, it can be seen that $D_N(x)$ is positive and greater in magnitude than $D_N(x+\frac{\pi}{N+1/2})$ for each x in the interval of integration, ultimately following from the fact that $k < N/2$ and that $\sin(x)$ is a strictly increasing function on $[0, \pi/2]$. This implies that the integrand of the above integral is always positive. Further, since f^* is monotone increasing and therefore always nonnegative, we have that

$$J_2 = |J_2| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} f^*\Big(\tfrac{(2k+2)\pi}{N+1/2}\Big) \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \big(D_N(x) + D_N(x+\tfrac{\pi}{N+1/2})\big)dx$$

Now, since this integral is equivalent to the one bounded in Lemma 1.6, we have that

$$|J_2| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} f^*\Big(\tfrac{(2k+2)\pi}{N+1/2}\Big)\Theta\Big(\frac{1}{k^2}\Big)$$

This time, we may use Lemma 1.4 - since f is Riemann integrable (and hence bounded), and the sum $\sum_{k\geq 1}\frac{1}{k^2}$ converges absolutely, we have that the above sum approaches zero, and $|J_2| \to 0$.

Thus, we have shown that both $|J_1|$ and $|J_2|$ approach zero as $N \to \infty$, implying that $|I_+| \to 0$ as well, as desired. Now, if f were monotone decreasing function on $[0, \epsilon]$ instead of monotone increasing, we could use the same argument to show that the partial Fourier sums of $-f$, a monotone increasing function, tend to zero. Since $S_N(-f) = -S_N f$, this would imply that $S_N f \to 0$ as well. Hence, the result holds for both monotone increasing and monotone decreasing functions on $[0, \epsilon]$ for any $\epsilon > 0$. Similarly, to show that $|I_-| \to 0$, we may note that the integral I_- is equivalent to the integral I_+ with $f(x)$ replaced by $f(-x)$. Since monotonicity of $f(x)$ on $[0, \epsilon]$ is equivalent to monotonicity of $f(-x)$ on $[-\epsilon, 0]$, we may use the same argument again to show that $|I_-|$ tends to zero as $N \to \infty$.

Finally, this allows us to conclude that for any function $f : \mathbb{T} \to \mathbb{R}$ that is monotone on $[0, \epsilon]$ and $[-\epsilon, 0]$ for some $\epsilon > 0$, and that further vanishes and is continuous at $x = 0$, its partial Fourier sums $S_N f$ must converge to 0 as $N \to \infty$. □

Corollary 6.6. *If $f : \mathbb{T} \to \mathbb{R}$ is Riemann integrable, continuous at $a \in \mathbb{T}$, and monotone in the intervals $[a, a+\epsilon]$ and $[a-\epsilon, a]$ for some $\epsilon > 0$, then the partial Fourier sums $S_N f(a) \to f(a)$ as $N \to \infty$.*

Proof. The function $g(x) = f(x+a) - f(a)$ satisfies the hypothesis of Theorem 1.4, and $S_N g(0) = S_N f(a) - f(a)$, implying that $S_N f(a) - f(a)$ tends to zero, and $S_N f(a) \to f(a)$ as $N \to \infty$. □

Theorem 6.7. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Riemann integrable function with $f(0) = 0$ and which satisfies the following "logarithmic continuity condition": for all x in some neighborhood of*

0,

$$|f(x+h)-f(x)| = o\Big(\frac{1}{\log h}\Big)$$

uniformly as $h \to 0$, so that the constants involved in the $o(\cdot)$ do not depend on x. Then the partial Fourier sums $S_N f(0) \to 0$ as $N \to \infty$.

Proof. Suppose that the function $f : \mathbb{T} \to \mathbb{R}$ satisfies $f(0) = 0$ as well as the following continuity condition for x in the neighborhood $(-\epsilon, \epsilon)$ for some $\epsilon > 0$:

$$|f(x+h)-f(x)| = o\Big(\frac{1}{\log h}\Big)$$

as $h \to 0$. In particular, by letting $x = 0$, it follows that we may choose an even smaller neighborhood $(-\epsilon', \epsilon') \subset (-\epsilon, \epsilon)$ with $0 < \epsilon' < \epsilon$ such that f satisfies

$$|f(h)| \leq -\frac{C}{\log h}$$

for some constant $C \geq 0$ and all $h \in (-\epsilon', \epsilon)$. Now, let us define the function f^* as follows:

$$f^*(x) = \begin{cases} f(-\epsilon') & x \in (-\pi, -\epsilon'] \\ f(x) & x \in (-\epsilon', \epsilon') \\ f(\epsilon') & x \in [\epsilon', \pi] \end{cases}$$

Since f satisfies the logarithmic continuity condition in the problem statement for $x \in (-\epsilon', \epsilon')$, and f^* is equal to f on this interval and constant everywhere else, it follows trivially that f^* satisfies the continuity condition for all $x \in \mathbb{T} \backslash \{\pi\}$.

Following the same vein as in the proof of Theorem 1.7, we may say by concentration of mass of the Dirichlet Kernel that

$$\int_0^\pi f(x) D_N(x) dx = \int_0^\pi f^*(x) D_N(x) dx + o(1)$$

or that $S_N f(0) = S_N f^*(0) + o(1)$, so it suffices to show that $S_N f^*(0) \to 0$ as $N \to \infty$.

If we proceed as in Theorem 1.4 and define $|I_+|$, $|I_-|$, $|J_1|$, and $|J_2|$ identically, we obtain the bounds

$$|J_1| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big| f^*(x) - f^*(x + \tfrac{\pi}{N+1/2}) \Big| \big| D_N(x) \big| dx$$

$$|J_2| \leq \sum_{k=0}^{\lfloor N/2 \rfloor - 1} \int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} \Big| f^*(x + \tfrac{\pi}{N+1/2}) \Big| \big(D_N(x) + D_N(x + \tfrac{\pi}{N+1/2}) \big) dx$$

Let us start by considering the bound on $|J_1|$. From the logarithmic continuity condition, we have that

$$\Big|f^*(x)-f^*(x+\tfrac{\pi}{N+1/2})\Big|=o\Big(\frac{1}{\log N}\Big)$$

for $x\in(-\pi,\pi)$. Hence, we have

$$\begin{aligned}
|J_1| &\le \sum_{k=0}^{\lfloor N/2\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}} o\Big(\frac{1}{\log N}\Big)\big|D_N(x)\big|dx\\
&= o\Big(\frac{1}{\log N}\Big)\sum_{k=0}^{\lfloor N/2\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}}\big|D_N(x)\big|dx\\
&= o\Big(\frac{1}{\log N}\Big)\sum_{k=0}^{\lfloor N/2\rfloor-1}\Theta\Big(\frac{1}{k}\Big)\\
&= o\Big(\frac{1}{\log N}\Big)\Theta\big(\log N\big)\\
&= o(1)
\end{aligned}$$

and so $|J_1|\to 0$, where we have used the bounds on the "hump areas" of the Dirichlet Kernel derived in Lemma 1.4.

Next we consider the bounds on $|J_2|$. First of all, leveraging concentration of mass of the Dirichlet Kernel again, we need only consider the first $\lfloor N\epsilon'/2\pi\rfloor-1$ terms of this sum of integrals:

$$|J_2|\le o(1)+\sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}}\Big|f^*(x+\tfrac{\pi}{N+1/2})\Big|\big(D_N(x)+D_N(x+\tfrac{\pi}{N+1/2})\big)dx$$

This restriction guarantees that $x\in(-\epsilon',\epsilon)$ for all x in the interval of integration of each integral in the sum. Hence, we may apply the logarithmic bounds on f^*:

$$\begin{aligned}
|J_2| &\le o(1)+\sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}}\Big|f^*(x+\tfrac{\pi}{N+1/2})\Big|\big(D_N(x)+D_N(x+\tfrac{\pi}{N+1/2})\big)dx\\
&\le o(1)-\sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor-1}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}}\frac{C}{\ln(\frac{(2k+2)\pi}{N+1/2})}\big(D_N(x)+D_N(x+\tfrac{\pi}{N+1/2})\big)dx\\
&\le o(1)-\sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor-1}\frac{C}{\ln(x+\frac{\pi}{N+1/2})}\int_{\frac{2k\pi}{N+1/2}}^{\frac{(2k+1)\pi}{N+1/2}}\big(D_N(x)+D_N(x+\tfrac{\pi}{N+1/2})\big)dx\\
&= o(1)-\sum_{k=0}^{\lfloor N\epsilon'/2\pi\rfloor-1}\frac{C}{\ln(\frac{(2k+2)\pi}{N+1/2})}\Theta\Big(\frac{1}{k^2}\Big)
\end{aligned}$$

which, again, is $o(1)$ as $N \to \infty$, meaning that $|J_2| \to 0$. Hence, we have that $|I_+| \to 0$ as $N \to \infty$ as well, and we may similarly argue that $|I_-| \to 0$ as $N \to \infty$ by considering the function $f(-x)$, which satisfies the same logarithmic continuity condition. Hence, we have the desired result: that $S_N f(0) \to 0$ as $N \to \infty$. $\square$

Corollary 6.8. *If $f : \mathbb{T} \to \mathbb{R}$ is Riemann integrable and satisfies the logarithmic continuity condition*

$$|f(x+h) - f(x)| = o\Big(\frac{1}{\log h}\Big)$$

uniformly as $h \to 0$ for all x in some neighborhood of $x = a$, then $S_N f(a) \to f(a)$ as $N \to \infty$.

Proof. The function $g(x) = f(x+a) - f(a)$ satisfies the hypothesis of Theorem 1.6, and $S_N g(0) = S_N f(a) - f(a)$, implying that $S_N f(a) - f(a) \to 0$ as $N \to \infty$, and hence $S_N f(a) \to f(a)$ as desired. $\square$

Corollary 6.9. *If $f : \mathbb{T} \to \mathbb{R}$ is Lipschitz or Hölder continuous everywhere on its domain, then $S_N f \to f$ uniformly on $\mathbb{T}$ as $N \to \infty$.*

Proof. Functions that are Lipschitz or Hölder everywhere satisfy the aforementioned logarithmic continuity condition trivially.

Corollary 6.10. *If $f : \mathbb{T} \to \mathbb{R}$ is differentiable everywhere on its domain, then $S_N f \to f$ uniformly on $\mathbb{T}$ as $N \to \infty$.*

Proof. Functions that are differentiable everywhere on $\mathbb{T}$ must also be locally Lipschitz at each point, and hence Lipschitz everywhere on $\mathbb{T}$ since it is a compact domain.

Other results (which I suspect to be stronger) have been proven guaranteeing the convergence of certain classes of continuous functions. In particular, Jordan has proven that the Fourier series of a function $f : \mathbb{T} \to \mathbb{R}$ of bounded variation converges to

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h)}{2}$$

from which it is an easy corollary that the Fourier series of a continuous functions of bounded variation converges uniformly to $f(x)$ at each $x \in \mathbb{T}$. [2] This was proven in 1881, but it happens that DuBois-Reymond actually mentions a criterion equivalent to the bounded-variation property in his 1873 paper:

> ...sie lautet, die Formel F gilt, wenn
>
> $$\int_0^a f'(\alpha) d\alpha$$
>
> absolut convergent ist.

References

[1] Paul DuBois-Reymond. "Ueber die Fourierschen Reihen". In: *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen* 1873.21 (Aug. 6, 1873), pp. 571–584. URL: https://gdz.sub.uni-goettingen.de/id/PPN252457072_1873?tify=%7B%22pages%22:[574],%22panX%22:0.336,%22panY%22:0.814,%22view%22:%22toc%22,%22zoom%22:0.382%7D (visited on 12/08/2021).

[2] María Cristina Pereyra and Lesley A. Ward. *Harmonic analysis: from Fourier to wavelets.* Student mathematical library 63. IAS/Park City mathematical subseries. Providence, R.I. : Princeton, N.J: American Mathematical Society ; Institute for Advanced Study, 2012. 410 pp. ISBN: 978-0-8218-7566-7.