Algebra of growth orders

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1. Defining growth orders

1.1. Definition of a growth order

The class of sequences that we will deal with shall be denoted $\mathcal{S}(\mathbb{R}^+)$, the set of sequences of positive real numbers. Notice that this is not what would traditionally be called a sequence space, because it is *not even a vector space.* It comes with operations of addition and scalar multiplication, but the underlying set of scalars $\mathbb{R}^+$ is not a field, as it lacks a zero element and additive inverses. We've sacrificed these properties for a reason: growth orders are meant to formalize the notion of asymptotic relative growth of sequences, and the negative real numbers and 0 aren't amenable to the concept of "relative size".

Definition 1. *Let $\alpha = (a_n)$ and $\beta = (b_n)$ be two sequences in $\mathcal{S}(\mathbb{R}^+)$. We will say that α, β* ***have the same growth order****, or $\alpha \sim \beta$, if there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that*

$$C_1 b_n \le a_n \le C_2 b_n$$

for all $n \in \mathbb{N}$.

There are several equivalent ways of thinking of this definition. The statement $\alpha \sim \beta$ is easily shown to be equivalent to each of the following:

- Both of the quotients a_n/b_n and b_n/a_n are bounded above in $\mathbb{R}^+$.
- The quotient a_n/b_n is bounded above and below by two strictly positive constants.
- $a_n = \Theta(b_n)$, or equivalently $b_n = \Theta(a_n)$, for those two are familiar with asymptotic notation.
- Both $a_n = O(b_n)$ and $b_n = O(a_n)$.
- The following limits are finite and positive:
$$\limsup_{n\to\infty} \frac{a_n}{b_n} \qquad \liminf_{n\to\infty} \frac{a_n}{b_n}$$

Now we shall prove that $\sim$ defines an *equivalence relation* on $\mathcal{S}(\mathbb{R}^+)$. The equivalence classes, consisting of all sequences with the same growth order, will be the objects that we refer to as *growth orders.*

Proposition 1. *The relation $\sim$ on sequences in $\mathcal{S}(\mathbb{R}^+)$ is an equivalence relation.*

Proof. Reflexivity follows trivially from the definition, taking $C_1 = C_2 = 1$ so that $C_1 a_n \le a_n \le C_2 a_n$ for all $n \in \mathbb{N}$, and $(a_n) \sim (a_n)$ for all sequences $(a_n) \in \mathcal{S}(\mathbb{R}^+)$.

To show symmetricity, suppose that $(a_n) \sim (b_n)$ for some sequences $(a_n), (b_n) \in \mathcal{S}(\mathbb{R}^+)$. Then we have that

$$C_1 b_n \le a_n \le C_2 b_n$$

for some $C_1, C_2 \in \mathbb{R}^+$. This implies, however, that

$$C_2^{-1} a_n \le b_n \le C_1^{-1} a_n$$

and therefore $(b_n) \sim (a_n)$, as desired.

Finally, to show transitivity, suppose that $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$ for some sequences $(a_n), (b_n), (c_n) \in \mathcal{S}(\mathbb{R}^+)$. Then we have that

$$C_1 b_n \le a_n \le C_2 b_n$$

$$C_3 c_n \le b_n \le C_4 c_n$$

for some constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$. This implies, however, that

$$C_1 C_3 c_n \le a_n \le C_2 C_4 c_n$$

and therefore $(a_n) \sim (c_n)$, so that transitivity holds as claimed. □

Now we are prepared to define growth orders as *equivalence classes*:

> **Definition 2.** *A* ***growth order*** *is defined as an equivalence class that is an element of $\mathcal{S}(\mathbb{R}^+)/\sim$. If $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$, then the* ***growth order of*** *α is the equivalence class of α under $\sim$, and may be denoted $[\alpha]$ or $[a_n]$.*

We will often use the font $\mathfrak{a}$ to refer to growth orders, so that Latin letters like a_n will refer to elements of sequences, Greek letters α will refer to sequences, and old German letters like $\mathfrak{a}$ will refer to equivalence classes of sequences comprising growth orders.

Exercise 1 Show that there are uncountably many distinct growth orders.

Exercise 2 Show that each growth order contains uncountably many sequences.

Exercise 3 Let $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$ be sequences of arbitrary growth orders. Determine if each statement is true or false in general.

1. If $a_n = b_n$ for all but finitely many values of n, then $(a_n) \sim (b_n)$.
2. If $|a_n - b_n| \to 0$ as $n \to \infty$, then $(a_n) \sim (b_n)$.
3. If $a_n / b_n \to 1$ as $n \to \infty$, then $(a_n) \sim (b_n)$.

Exercise 4 Show that, for any sequence (a_n) in a growth order $\mathfrak{a}$, there exists a sequence $(a'_n) \in \mathfrak{a}$ such that $a_n \le a'_n$ for all $n \in \mathbb{N}$.

1.2. Common growth orders

This is just a short section meant to establish notation that we will use later in the write-up to denote some commonly-occurring growth orders.

- $\mathfrak{1}$ denotes the constant growth order $[(1)]$.
- $\mathfrak{n}$ denotes the growth order $[(n)]$.
- $\mathfrak{n}^p$ denotes the growth order $[(n^p)]$, for $p \in \mathbb{R}^+$.
- $\mathfrak{u}$ denotes the pathological growth order $[(n^{(-1)^n})]$, which we will often use as a counterexample because of its oscillatory nature.
- $\mathfrak{l}$ denotes the growth order $[(\log n)]$.
- $\mathfrak{l}_m$ denotes the growth order $[(\log \cdots \log n)]$, where there are m nested logs.
- $\mathfrak{l}(p_0, p_1, \cdots, p_m)$ denotes the growth order of the sequence

$$n^{p_0}(\log n)^{p_1} \cdots (\overbrace{\log \cdots \log}^{m \text{ nested logs}} n)^{p_m}$$

  which, of course, must be defined for $n \geq \lceil {}^m e \rceil$, where the superscript denotes tetration here.

1.3. Moderate growth orders

Now that we've defined growth orders in general, we'll define a certain class of sequences that we will pay particular attention to.

Definition 3. *Let us call $\alpha \in \mathcal{S}(\mathbb{R}^+)$ a sequence of* ***moderate growth*** *if, for any $k \in \mathbb{N}$, there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that*

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m \leq kn$.

This "niceness" condition will become useful later, for defining operations such as the composition of two sequences. Although it is not immediately obvious from the definition, moderate growth can be equated, in some weak sense, with polynomial growth. To be precise, every moderate sequence that is bounded between power-sequences of the form (Cn^p) with $p \in \mathbb{R}$ and $C \in \mathbb{R}^+$,.

Proposition 2. *If a sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$ exhibits moderate growth, then there exist $p, q \in \mathbb{R}$ and $C_1, C_2 \in \mathbb{R}^+$ such that*

$$C_1 n^p \leq a_n \leq C_2 n^q$$

for all $n \in \mathbb{N}$. The converse is not true, but any sequence for which the above inequality holds with $p = q$ necessarily has moderate growth.

Proof. Suppose that (a_n) exhibits moderate growth. Then let C_1, C_2 be constants such that

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n \leq m \leq 2n$. Inductively, we may show that for any $n \in \mathbb{N}$,

$$a_n \leq C_2 a_{\lceil n/2 \rceil} \leq C_2^2 a_{\lceil n/4 \rceil} \leq \cdots \leq C_2^{\lceil \log_2 n \rceil} a_1 \leq a_1 \max(1, C_2) \cdot n^{\log_2 C_2}$$

and similarly

$$a_n \geq C_1 a_{\lceil n/2 \rceil} \geq C_1^2 a_{\lceil n/4 \rceil} \geq \cdots \geq C_1^{\lceil \log_2 n \rceil} a_1 \geq a_1 \min(1, C_1) \cdot n^{\log_2 C_1}$$

and so we have

$$a_1 \min(1, C_1) \cdot n^{\log_2 C_1} \leq a_n \leq a_1 \max(1, C_2) \cdot n^{\log_2 C_2}$$

which proves the first claim, taking $p = \log_2 C_1$ and $q = \log_2 C_2$.

To see why the converse is not true, consider, for instance, the sequence $a_n = n^{(-1)^n}$. For all odd $m \in \mathbb{N}$, we have $a_m = 1/m$, whereas $a_{2m} = m$, meaning that $a_{2m}/a_m = m^2$ is unbounded, and (a_n) does not satisfy the moderate growth property.

However, if we are given that

$$C_1 n^p \leq a_n \leq C_2 n^p$$

where $p \in \mathbb{R}$ and $C_1, C_2 \in \mathbb{R}^+$ (the stronger special case in which $p = q$), moderate growth follows. For we have that if $n \leq m \leq kn$ for some fixed $k \in \mathbb{N}$, then

$$a_m \leq C_2 m^p \leq C_2 \max(1, k^p) n^p \leq \frac{C_2 \max(1, k^p)}{C_1} a_n$$

and

$$a_m \geq C_1 m^p \geq C_1 \min(1, k^p) n^p \geq \frac{C_1 \min(1, k^p)}{C_2} a_n$$

so that we have

$$\frac{C_1 \min(1, k^p)}{C_2} a_n \leq a_m \leq \frac{C_2 \max(1, k^p)}{C_1} a_n$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ with $n \leq m \leq kn$, so that it follows that (a_n) has moderate growth by definition. □

Here are some propositions that provide sufficient (equivalent) conditions for moderacy that have less stringent requirements, and are therefore easier to prove for some sequences.

Proposition 3. *In order for $\alpha \in \mathcal{S}(\mathbb{R}^+)$ to exhibit moderate growth, it is sufficient for there to exist $C_1, C_2 \in \mathbb{R}^+$ such that*

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m \leq 2n$. In other words, it suffices to find such constants for the case of $k = 2$ in the definition of moderate growth.

Proof. Suppose that such constants $C_1, C_2 > 0$ exist for $k = 2$, and suppose WLOG that $C_1 < 1$ and $C_2 > 1$ (for if not, we may simply decrease C_1 below 1 and increase C_2 above 1, weakening the inequality). Then we may show by induction that for all $q \in \mathbb{N}$, and for all $m, n \in \mathbb{N}$ with $n \leq m \leq 2^q n$, the following inequality holds:

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

As our inductive hypothesis, we assume that this holds for some value of q. Then we clearly have that

$$C_1 a_{2^q n} \leq a_m \leq C_2 a_{2^q n}$$

for all $2^q n \leq m \leq 2^{q+1} n$ which is a direct consequence of our original assumption, in which n is replaced by $2^q n$. But since $C_1^q a_n \leq a_{2^q n} \leq C_2^q a_n$ by the inductive hypothesis, we have that

$$C_1^{q+1} a_n \leq C_1 a_{2^q n} \leq a_m \leq C_2 a_{2^q n} \leq C_2^{q+1} a_n$$

and thus

$$C_1^{q+1} a_n \leq a_m \leq C_2^{q+1} a_n$$

for all $2^q n \leq m \leq 2^{q+1} n$. Since the tighter inequality

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

holds for $n \leq m \leq 2^q n$ by the inductive hypothesis, we may combine the two cases of $n \leq m \leq 2^q n$ and $2^q n \leq m \leq 2^{q+1} n$ and state that

$$C_1^{q+1} a_n \leq a_m \leq C_2^{q+1} a_n$$

for all $n \leq m \leq 2^{q+1} n$.

Thus, the truth of our inequality for some $q \in \mathbb{N}$ implies its truth for $q + 1$. But the base case of $q = 1$ is taken as an assumption, so we have by induction that for all $q \in \mathbb{N}$ and $n \leq m \leq 2^q n$ the inequality

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

holds. Since, for all $k \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that $2^q \geq k$, if some $k \in \mathbb{N}$ is given, we may choose such a value of $q \in \mathbb{N}$, and then bound

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

for all $n \leq m \leq kn \leq 2^q n$, demonstrating that the sequence (a_n) is moderate by definition. □

Proposition 4. *In order for $\alpha \in \mathcal{S}(\mathbb{R}^+)$ to exhibit moderate growth, it is sufficient for there to exist $C_1, C_2 \in \mathbb{R}^+$ such that*

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m \leq \rceil rn \lceil$ for some $r > 1$. This is a further weakening of the above proposition.

Proof. Using a similar argument as shown in the above proof, we may show that if this is true for some $r > 1$ with $C_1 < 1$ and $C_2 > 1$, then it follows that

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

for all $n \leq m \leq \lceil r^q n \rceil$, again using a proof by induction. Because $r > 1$, there exists $q \in \mathbb{N}$ such that $r^q \geq 2$, so that we have

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

for all $n \leq m \leq 2n \leq \lceil r^q n \rceil$. The result then follows from the previous proposition. □

We may also refer to *growth orders* as being moderate, depending on whether or not they consist of sequences of moderate growth. Next up we will prove that no equivalence class contains both moderate and non-moderate sequences, meaning that it makes sense to say that a growth order is moderate or non-moderate.

Proposition 5. *Let $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha] = [\alpha']$. Then α exhibits moderate growth iff α' exhibits moderate growth.*

Proof. Suppose $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha] = [\alpha']$, so that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

for all $n \in \mathbb{N}$ for some $C_1, C_2 \in \mathbb{R}^+$. Suppose WLOG that α exhibits moderate growth, so that for each $k \in \mathbb{N}$, there exist constants $C_3, C_4 \in \mathbb{R}^+$ such that

$$C_3 a_n \leq a_m \leq C_4 a_n$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ between n and kn. Then, if $k \in \mathbb{N}$ is fixed, and $n \leq m \leq kn$ for some $m, n \in \mathbb{N}$, we have

$$a'_m \leq C_2 a_m \leq C_2 C_4 a_n \leq \frac{C_2 C_4}{C_1} a'_n$$

and

$$a'_m \geq C_1 a_m \geq C_1 C_3 a_n \geq \frac{C_1 C_3}{C_2} a'_n$$

so we have that

$$\frac{C_1 C_3}{C_2} a'_n \leq a'_m \leq \frac{C_2 C_4}{C_1} a'_n$$

and therefore α' has moderate growth. Thus, moderate growth of α implies moderate growth of α' and vice versa (by symmetry). □

The following definition is therefore justified:

Definition 4. *A growth order* $\mathfrak{a}$ *is said to be* ***moderate*** *if each of its sequences has moderate growth, and* ***not moderate*** *if none of its sequences has moderate growth.*

Moderate growth sequences have some convenient properties that we'll really come to appreciate when it's time to define the composition operation in a later section. For now, however, we can state and prove a few of their elementary properties.

Proposition 6. *If* $\alpha = (a_n)$ *is a moderate growth sequence, then every arithmetic subsequence* (a_{jn+k}) *with* $j, k \in \mathbb{N}$ *has the same growth order.*

Proof. If $j, k, n \in \mathbb{N}$, then we have $n \leq jn + k \leq (j + k)n$, so we have $(a_n) \sim (a_{jn+k})$ by the moderate growth property of α. □

What sorts of horrible sequences *do not* have this property, you might ask? One example is the pathological sequence $a_n = n^{(-1)^n}$ that was used as a counterexample earlier. However, there are also many naturally-occurring sequences without this property, such as exponential sequences like $a_n = 2^n$, for which $(a_{2n}) > (a_n)$.

This gives rise to an important notational issue. Later on in this write-up, I may refer to sequences such as $(\log n)$ or $(\log \log n)$ which, strictly speaking, aren't members of $\mathcal{S}(\mathbb{R}^+)$ - in the former case because $\log 1 = 0$ is not a positive number, and in the latter case because $\log \log 1 = \log 0$ does not exist. However, if we are dealing with moderate growth sequences, the shifted sequence (a_{n+1}) has the same growth order as the original sequence (a_n). That is *moderate growth orders are translation-invariant.* Hence, for sequences like $(\log \log n)$, we can simply shift the sequence over finitely many terms in order to start at the first term for which it is defined and in $\mathbb{R}^+$. In this case, the sequence remains in $\mathbb{R}^+$ for $n \geq 3$. So when we write $(\log \log n)$, we are really referring to the sequence $(\log \log(n + 3)) \in \mathcal{S}(\mathbb{R}^+)$.

There are, of course, sequences that are not translation-invariant. The classic pathological example $a_n = n^{(-1)^n}$ works here as well, but another example that feels less contrived is the sequence $a_n = 2^{n^2}$.

Proposition 7. *If* $\alpha = (a_n) \in \mathfrak{a}$ *is moderate, then*

$$\left(\sum_{i=n}^{kn} a_i \right) \sim (n a_n)$$

for any $k \in \mathbb{N}$.

Proof. Given $k \in \mathbb{N}$, if (a_n) is moderate, then we have constants C_1, C_2 such that $C_1 a_n \leq a_m \leq$

$C_2 a_n$ for all m with $n \leq m \leq kn$. Thus, we have that

$$\sum_{i=n}^{kn} a_i \leq C_2 \sum_{i=n}^{kn} a_n = C_2(kn - n + 1)a_n$$

and this upper bound is, of course, $\sim (na_n)$. On the other hand, we also have that

$$\sum_{i=n}^{kn} a_i \geq C_1 \sum_{i=n}^{kn} a_n = C_1(kn - n + 1)a_n$$

so we also have a lower bound that is $\sim (na_n)$. Hence, we have that

$$\left(\sum_{i=n}^{kn} a_i \right) \sim (na_n)$$

as claimed. □

Simply knowing that a sequence is moderate gives us an easy shortcut for evaluating sums of the above form - simply multiply them by $\mathfrak{n}$! This allows us to immediately deduce asymptotic formulas such as the following:

$$\sum_{k=n}^{2n} \frac{\log^2 k}{k} = \Theta(\log^2 n)$$

...provided, of course, that $(\log^2 n/n)$ is a moderate sequence.

2. Partial ordering

2.1. Definition of the ordering

Definition 5. *Let $\mathfrak{a}, \mathfrak{b}$ be growth orders. We will say that $\mathfrak{a} \leq \mathfrak{b}$, or $\mathfrak{a}$* ***grows at most as fast*** *as $\mathfrak{b}$, if, for each $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$, there exists a constant $C \in \mathbb{R}^+$ such that*

$$a_n \leq Cb_n$$

for all $n \in \mathbb{N}$. Further, we will say that $\mathfrak{a} < \mathfrak{b}$, or $\mathfrak{a}$ ***grows slower*** *than $\mathfrak{b}$, if $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{b}$.*

It is straightforward to show that the above defines a partial ordering on the growth orders over $\mathcal{S}(\mathbb{R}^+)$.

Proposition 8. *The above defines a partial ordering on growth orders in $\mathcal{S}(\mathbb{R}^+)/\sim$.*

Proof. We immediately have the reflexive property, namely that $\mathfrak{a} \leq \mathfrak{a}$ for all $\mathfrak{a}$, for if $(a_n) \in \mathfrak{a}$, we have that $a_n \leq Ca_n$ for all $n \in \mathbb{N}$ when $C = 1$.

To prove transitivity, let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be growth orders with $\mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{c}$. If $(a_n) \in \mathfrak{a}, (b_n) \in \mathfrak{b}$, and $(c_n) \in \mathfrak{c}$, then there exist constants C_1, C_2 such that $a_n \leq C_1 b_n$ and $b_n \leq C_2 c_n$, and therefore $a_n \leq C_1 C_2 c_n$ for all $n \in \mathbb{N}$.

Finally, we shall prove antisymmetry: namely that $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$ together imply that $\mathfrak{a} = \mathfrak{b}$. If both of these inequalities hold, then for all $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$, there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that $a_n \leq C_1 b_n$ and $b_n \leq C_2 a_n$, meaning that

$$C_1^{-1} a_n \leq b_n \leq C_2 a_n$$

and therefore $(a_n) \sim (b_n)$ and $\mathfrak{a} = \mathfrak{b}$. ◻

Proposition 9. *The following are equivalent to $\alpha \leq \beta$:*

- $a_n = O(b_n)$
- a_n / b_n *is bounded*
- $a_n / b_n \leq (1)$

Proof. Follows directly from definitions. ◻

2.2. Chains and antichains

This defines a partial ordering on $\mathcal{S}(\mathbb{R}^+)/\sim$, but it is *not* a total order. That is, trichotomy does not hold, and there exist growth orders $\mathfrak{a}$ and $\mathfrak{b}$ such that neither $\mathfrak{a} \leq \mathfrak{b}$ not $\mathfrak{b} \leq \mathfrak{a}$. For instance, consider $\mathfrak{a} = \mathbf{1}$ and $\mathfrak{b} = \mathfrak{u}$.

Definition 6. *If $\mathfrak{a} \nleq \mathfrak{b}$ and $\mathfrak{b} \nleq \mathfrak{a}$, then we say that $\mathfrak{a}$ is **incomparable** to $\mathfrak{b}$, and write $\mathfrak{a} \perp \mathfrak{b}$. A **chain** is a set of growth orders of which any two are comparable, and an **antichain** is a set of growth orders of which any two are incomparable.*

In the future, we would like to work with *chains* of growth orders when possible, since it is most convenient for arithmetic to have any two growth orders be comparable. However, it is harder than it seems to concisely describe a way of restricting $\mathcal{S}(\mathbb{R}^+)/\sim$ to a subset of growth orders that is both closed under desirable operations (to be introduced later) while still possessing trichotomy. Here are a few fairly well-behaved chains of $\mathcal{S}(\mathbb{R}^+)/\sim$:

- The set of polynomial growth orders $\mathfrak{n}^p$ with $p \in \mathbb{N}$, which has order type ω.
- The set of power-function growth orders $\mathfrak{n}^p$ with $p \in \mathbb{R}$, which has order type λ (the order type of $\mathbb{R}$).
- The set of growth orders taking the form $\mathfrak{n}^p\mathfrak{l}^q = [(n^p \log^q n)]$ with $p, q \in \mathbb{R}$. This has order type λ^2.

This gives rise to the following question:

Question 1 Which ordinal numbers are the order type of some chain of $\mathcal{S}(\mathbb{R}^+)/\sim$? That is, what is the smallest ordinal that *cannot* be embedded in $\mathcal{S}(\mathbb{R}^+)/\sim$?

It happens that $\mathcal{S}(\mathbb{R}^+)/\sim$ also has some very large antichains. For instance, consider the family of growth orders

$$\mathfrak{a}_p = n^{p(-1)^n}$$

where $p \in \mathbb{R}$. We can see that $\mathfrak{a}_p \perp \mathfrak{a}_q$ for all $p \neq q$, for the ratio of the terms $n^{p(-1)^n}$ and $n^{q(-1)^n}$ will oscillate between very large and very small values. The above observations prove the following proposition:

Proposition 10. *$\mathcal{S}(\mathbb{R}^+)/\sim$ has both uncountable chains and uncountable antichains.*

3. Arithmetic

3.1. Sums, products and quotients

In this section, we will prove that the elementwise arithmetic operations of $+, \cdot, \div$ on sequences in $\mathcal{S}(\mathbb{R}^+)$ can be extended to growth orders in the obvious way without accidentally introducing any ill-founded expressions. Let us first define these operations on sequences, and then extend the definition to growth orders:

Definition 7. *Given $\alpha = (a_n), \beta = (b_n) \in \mathcal{S}(\mathbb{R}^+)$, define their* ***elementwise sum*** *$\alpha + \beta = (a_n + b_n)$, their* ***elementwise product*** *$\alpha \cdot \beta = (a_n b_n)$, and their* ***elementwise quotient*** *$\alpha/\beta = (a_n/b_n)$. The* ***elementwise reciprocal*** *of α is defined as $\alpha^{-1} = (a_n^{-1})$.*

Proposition 11. *If $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$, then $[\alpha + \beta] = [\alpha' + \beta']$.*

Proof. Let $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$. Then there exist constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

$$C_3 b_n \leq b'_n \leq C_4 b_n$$

for all $n \in \mathbb{N}$. By adding these inequalities, we have that

$$C_1 a_n + C_3 b_n \leq a'_n + b'_n \leq C_2 a_n + C_4 b_n$$

and, since a_n, b_n are positive integers, we have

$$\min(C_1, C_3)(a_n + b_n) \leq a'_n + b'_n \leq \max(C_2, C_4)(a_n + b_n)$$

and therefore $\alpha + \beta \sim \alpha' + \beta'$, proving the claim. $\square$

Proposition 12. *If $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$, then $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$.*

Proof. Let $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$. Then there exist constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

$$C_3 b_n \leq b'_n \leq C_4 b_n$$

for all $n \in \mathbb{N}$. Multiplying these inequalities yields

$$C_1 C_3 a_n b_n \leq a'_n b'_n \leq C_2 C_4 a_n b_n$$

so that we immediately have $\alpha \cdot \beta \sim \alpha' \cdot \beta'$. □

Proposition 13. *If $\alpha, \alpha' \in \mathfrak{a}$, we have $[\alpha^{-1}] = [\alpha'^{-1}]$.*

Proof. Let $\alpha, \alpha' \in \mathfrak{a}$ so that there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

for all $n \in \mathbb{N}$. Inverting these inequalities gives the inequalities

$$C_2{}^{-1} a_n{}^{-1} \leq a'_n{}^{-1} \leq C_1{}^{-1} a_n{}^{-1}$$

so that $\alpha^{-1} \sim \alpha'^{-1}$ as desired. □

This means that the growth orders given by $[\alpha + \beta]$, $[\alpha \cdot \beta]$, and $[\alpha^{-1}]$ depend only on the growth orders of α and β, and may as well be defined as functions of $\mathfrak{a}$ and $\mathfrak{b}$. This leads to the next definition:

Definition 8. *Given growth orders $\mathfrak{a} = [\alpha]$ and $\mathfrak{b} = [\beta]$, define their* ***sum*** *$\mathfrak{a} + \mathfrak{b} = [\alpha + \beta]$, their* ***product*** *$\mathfrak{a} \cdot \mathfrak{b} = [\alpha \cdot \beta]$, and their* ***quotient*** *$\mathfrak{a}/\mathfrak{b} = [\alpha/\beta] = [\alpha \cdot \beta^{-1}]$. Define the* ***reciprocal*** *of the growth order $\mathfrak{a}$ as $\mathfrak{a}^{-1} = [\alpha^{-1}] = [1/\alpha]$.*

From the definitions of elementwise addition, products, and quotients, the following familiar algebraic identities immediately follow:

Proposition 14. *For all growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we have the following identities:*
- $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$
- $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$
- $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$
- $(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c})$
- $\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{c}$
- $\mathfrak{a} \cdot \mathbf{1} = \mathfrak{a}$
- $\mathfrak{a} \cdot \mathfrak{a}^{-1} = \mathbf{1}$
- $\mathfrak{a} \cdot \mathfrak{b}^{-1} = \mathfrak{a}/\mathfrak{b}$

Proof. Trivial. Because the analogues of these identities hold for elements of $\mathbb{R}^+$, and these operations are defined elementwise on sequences, their analogues hold in $\mathcal{S}(\mathbb{R}^+)$. Since arithmetic on growth orders is define by the arithmetic on their constituent sequences, the above identities follow. □

3.2. Preserving moderate growth

In this section we will show briefly that these operations preserve the moderate growth property, so that we may freely take sums and products of moderate growth sequences without worrying about inadvertently producing immoderate growth sequences.

Proposition 15. *If $\mathfrak{a}, \mathfrak{b}$ are moderate growth sequences, then $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cdot \mathfrak{b}$ and $\mathfrak{a}^{-1}$ are moderate growth sequences.*

Proof. Let $\mathfrak{a}, \mathfrak{b}$ be moderate growth sequences, so that for all $k, m, n \in \mathbb{N}$ with $n \le m \le kn$, we have constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ such that

$$C_1 a_n \le a_m \le C_2 a_n$$

$$C_3 b_n \le b_m \le C_4 b_n$$

Then we have

$$\min(C_1, C_3)(a_n + b_n) \le a_m + b_m \le \max(C_2, C_4)(a_n + b_n)$$

so that $\mathfrak{a} + \mathfrak{b}$ has moderate growth. We also have

$$C_1 C_3 (a_n \cdot b_n) \le a_m \cdot b_m \le C_2 C_4 (a_n \cdot b_n)$$

so that $\mathfrak{a} \cdot \mathfrak{b}$ has moderate growth, and

$$C_2^{-1} a_n^{-1} \le a_m^{-1} \le C_1^{-1} a_n$$

so that $\mathfrak{a}^{-1}$ has moderate growth as well. □

3.3. Subtraction and exponentiation

All of the trouble we've gone to in the above sections to define the simple operations of addition, multiplication, and division might seem overly pedantic. After all, these operations extend to growth orders exactly how we'd expect them to, and their properties are more or less what we'd expect. So why go to all this trouble to show that they're well-defined? As it happens, not all operations from real arithmetic extend so nicely to $\mathcal{S}(\mathbb{R}^+)$, and in this section I'll briefly give two examples: subtraction and exponentiation.

After defining addition on growth orders, it seems a natural next step to attempt a definition of subtraction. Perhaps we could define $\mathfrak{a} - \mathfrak{b}$ as the growth order of the sequence $(a_n - b_n)$. An obvious issue is that the difference $a_n - b_n$ may be negative or zero, and therefore $\notin \mathbb{R}^+$. However, this could be remedied by considering instead the absolute difference $|a_n - b_n|$, or by letting $\mathfrak{a} - \mathfrak{b}$ be defined only when $\mathfrak{a} > \mathfrak{b}$.

However, this approach is not viable either. Consider the following three sequences:

$$a_n = n + \frac{1}{n} + \frac{1}{n^2}$$

$$b_n = n$$

$$b'_n = n + \frac{1}{n}$$

Then we have that $(b_n) \sim (b'_n)$, while $(a_n - b_n) \sim (1/n)$ and $(a_n - b'_n) \sim (1/n^2)$, which *do not* have the same growth order. That illustrates why the growth order of the difference $(a_n - b_n)$ *does not* depend only on the growth orders of (a_n) and (b_n), and therefore the difference $\mathfrak{a} - \mathfrak{b}$ cannot be well-defined.

In fact, $\mathcal{S}(\mathbb{R}^+)/\sim$ does not have a *cancellation law*, so it is impossible in principle to define an operation $-$ satisfying $(\mathfrak{a} + \mathfrak{b}) - \mathfrak{b} = \mathfrak{a}$ - which is something that we would certainly want subtraction to satisfy if we were to define it! To be explicit, if $\mathfrak{b}_1, \mathfrak{b}_2$ are two distinct growth orders both $\leq \mathfrak{a}$, then we would have $\mathfrak{a} + \mathfrak{b}_1 = \mathfrak{a} + \mathfrak{b}_2 = \mathfrak{a}$ (by a property proven in the next section) and therefore

$$\mathfrak{b}_1 = (\mathfrak{a} + \mathfrak{b}_1) - \mathfrak{a} = \mathfrak{a} - \mathfrak{a} = (\mathfrak{a} + \mathfrak{b}_2) - \mathfrak{a} = \mathfrak{b}_2$$

which is a contradiction! (The interesting question of whether subtraction or additive inverses could be reasonably extended to $\mathcal{S}(\mathbb{R}^+)/\sim$ was posed by Nic Berkopec.)

Exponentiation is another example: we cannot define $\mathfrak{a}^{\mathfrak{b}}$ as the growth order of $(a_n^{b_n})$, because this is not uniquely defined by the growth orders of (a_n) and (b_n). For instance, consider $a_n = 2$, $a'_n = 3$, $b_n = n$, and $b'_n = 2n$. Then the sequences $({a_n}^{b_n})$, $({a'_n}^{b_n})$, $({a_n}^{b'_n})$, and ${a'_n}^{b'_n}$, causing the desired property to fail catastrophically. We have

$$\begin{aligned} ({a_n}^{b_n}) &= (2^n) \\ ({a'_n}^{b_n}) &= (3^n) \\ ({a_n}^{b'_n}) &= (4^n) \\ ({a'_n}^{b'_n}) &= (9^n) \end{aligned}$$

so that

$$({a_n}^{b_n}) < ({a'_n}^{b_n}) < ({a_n}^{b'_n}) < ({a'_n}^{b'_n})$$

In general, for any growth orders $\mathfrak{a}, \mathfrak{b} > \mathbf{1}$, there are infinitely many different growth orders among the sequences $({a_n}^{b_n})$ with $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$.

3.4. Inequalities

Proposition 16. *We have* $\mathfrak{a} \leq \mathfrak{b}$ *iff* $\mathfrak{a}/\mathfrak{b} \leq \mathbf{1}$*, and* $\mathfrak{a} \perp \mathfrak{b}$ *iff* $\mathfrak{a}/\mathfrak{b} \perp \mathbf{1}$*.*

Proof. Follows straight from the definitions. Because we are dealing with sequences of positive real numbers, we have that $a_n \leq Cb_n$ iff $a_n/b_n \leq C \cdot 1$, from which $\mathfrak{a} \leq \mathfrak{b} \iff \mathfrak{a}/\mathfrak{b} \leq \mathbf{1}$ immediately follows. Since $\mathfrak{a} \perp \mathfrak{b}$ iff neither $\mathfrak{a} \leq \mathfrak{b}$ nor $\mathfrak{b} \leq \mathfrak{a}$, we have that $\mathfrak{a} \perp \mathfrak{b} \iff \mathfrak{a}/\mathfrak{b} \perp \mathbf{1}$ follows immediately. □

Proposition 17. *If* $\mathfrak{a}, \mathfrak{b}$ *are comparable, then* $\mathfrak{a} + \mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b})$*.*

Proof. Suppose WLOG that $\mathfrak{a} \geq \mathfrak{b}$. If $\alpha = (a_n) \in \mathfrak{a}$ and $\beta = (b_n) \in \mathfrak{b}$, we have that there exists a constant $C \in \mathbb{R}^+$ such that $a_n \geq Cb_n$ for all $n \in \mathbb{N}$, implying that $a_n \geq \frac{C}{C+1}(a_n + b_n)$ and therefore $\mathfrak{a} \geq \mathfrak{a} + \mathfrak{b}$. On the other hand, we have $a_n \leq a_n + b_n$, so $\mathfrak{a} \leq \mathfrak{a} + \mathfrak{b}$, and therefore $\mathfrak{a} = \max(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$. □

This might give the impression that + is a rather trivial operation on growth orders. However, the above only applies to comparable growth orders: the situation is more complicated (and interesting!) on incomparable growth orders $\mathfrak{a} \perp \mathfrak{b}$.

Proposition 18. *The set of growth orders* $\mathcal{S}(\mathbb{R}^+)/\sim$ *comprises a lattice in which the* ***join*** *and* ***meet*** *are respectively defined by*

$$\mathfrak{a} \vee \mathfrak{b} = \mathfrak{a} + \mathfrak{b}$$

$$\mathfrak{a} \wedge \mathfrak{b} = (\mathfrak{a}^{-1} + \mathfrak{b}^{-1})^{-1}$$

so that $\mathfrak{a} \vee \mathfrak{b}$ *is the unique least upper bound of* $\mathfrak{a}, \mathfrak{b}$*, and* $\mathfrak{a} \wedge \mathfrak{b}$ *is their unique greatest lower bound.*

Proof. First we prove that $\mathfrak{a} \vee \mathfrak{b}$ is the unique least upper bound of $\mathfrak{a}$ and $\mathfrak{b}$. Suppose that $\mathfrak{c} \geq \mathfrak{a}, \mathfrak{b}$, so that for all $(a_n) \in \mathfrak{a}, (b_n) \in \mathfrak{b}, (c_n) \in \mathfrak{c}$, we have constants $C_1, C_2 \in \mathbb{R}^+$ such that $a_n \leq C_1 c_n$ and $b_n \leq C_2 c_n$ for all $n \in \mathbb{N}$, and therefore $a_n + b_n \leq (C_1 + C_2)c_n$, meaning that $\mathfrak{a} + \mathfrak{b} \leq \mathfrak{c}$. Hence $\mathfrak{a} + \mathfrak{b}$ is a least upper bound for $\mathfrak{a}, \mathfrak{b}$ (because any other common upper bound $\mathfrak{c}$ must grow at least as fast as it does). Uniqueness follows from antisymmetry of $\leq$: if there were two least upper bounds $\mathfrak{c}_1, \mathfrak{c}_2$, we would have that $\mathfrak{c}_1 \leq \mathfrak{c}_2$ and $\mathfrak{c}_2 \leq \mathfrak{c}_1$, and therefore $\mathfrak{c}_1 = \mathfrak{c}_2$.

To show that $\mathfrak{a} \wedge \mathfrak{b}$ is the unique greatest lower bound, notice that $\mathfrak{c}$ is a lower bound for $\mathfrak{a}, \mathfrak{b}$ if and only if $\mathfrak{c}^{-1}$ is an upper bound for $\mathfrak{a}^{-1}, \mathfrak{b}^{-1}$ because of the decreasing nature of the function $\cdot \mapsto \cdot^{-1}$. Hence, the greatest-lower-bound property of $\mathfrak{a} \wedge \mathfrak{b}$, as well as its uniqueness, is a corollary of least-upper-bound property of $\mathfrak{a} \vee \mathfrak{b}$ combined with the fact that $\mathfrak{a} \vee \mathfrak{b} = (\mathfrak{a}^{-1} \wedge \mathfrak{b}^{-1})^{-1}$. □

The above proposition implies that every pair of growth orders has a *least upper bound* and a *greatest lower bound*, and consequently that any *finite* collection of growth orders has a LUB and a GLB (which can be formed by repeatedly taking pairwise LUBs and GLBs). A natural question to ask is whether *arbitrary* bounded sets of growth orders also have unique least upper bounds and greatest lower bounds. However, the question of whether least upper bounds exist in $\mathcal{S}(\mathbb{R}^+)$ can be answered in the negative fairly quickly. Consider, for instance, the chain

$$\mathfrak{n} < \mathfrak{n}^2 < \mathfrak{n}^3 < \cdots$$

and suppose that $\mathfrak{a}$ is an upper bound for the set $\{\mathfrak{n}^p\}_{p\in\mathbb{N}}$. No matter the value of $\mathfrak{a}$, there always exists a smaller upper bound for this chain. For instance, $\mathfrak{a}/\mathfrak{n}$ will suffice: if $\mathfrak{a} > \mathfrak{n}^p$ for all $p \in \mathbb{N}$, then $\mathfrak{a}/\mathfrak{n} > \mathfrak{n}^p$ for all $p \in \mathbb{N}$ as well, yet $\mathfrak{a}/\mathfrak{n} < \mathfrak{a}$.

Thus, we cannot even get least upper bounds for increasing sequences of growth orders in $\mathcal{S}(\mathbb{R}^+)$. A natural follow-up question is whether *any* strictly increasing sequence of growth orders has a least upper bound. That is, if we have an increasing sequence of growth orders

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots$$

can we *always* conclude that no upper bound is a least upper bound, as was the case with the chain $\mathfrak{n} < \mathfrak{n}^2 < \cdots$?

Proposition 19. *For any strictly increasing sequence of growth orders*

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots$$

with an upper bound $\mathfrak{a}' > \mathfrak{a}_i$ for all $i \in \mathbb{N}$, then there exists another upper bound $\mathfrak{b}$ with $\mathfrak{b} > \mathfrak{a}_i$ for all $i \in \mathbb{N}$ and $\mathfrak{b} < \mathfrak{a}'$.

Proof. Making use of the Axiom of Choice, we may consider some infinite sequence of sequences $(a_n^{(i)}) \in \mathfrak{a}_i$ for $i \in \mathbb{N}$. Without loss of generalize, we may assume that $a_n^{(i)} \leq a_n^{(j)}$ for all $i < j$ and all $n \in \mathbb{N}$. For if the sequences we choose do not satisfy these inequalities, we may let C_i be a family of constants such that $a_n^{(i)} \leq C_i a_n^{(i+1)}$ for all $n \in \mathbb{N}$ (since $\mathfrak{a}_i < \mathfrak{a}_{i+1}$) and replace the sequences $(a_n^{(1)}), (a_n^{(2)}), (a_n^{(3)}), \cdots$ with the sequences $(a_n^{(1)}), (C_1 a_n^{(2)}), (C_1 C_2 a_n^{(3)}), \cdots$, which have the same respective growth orders while satisfying the desired inequalities.

Having chosen a sequence of sequences $(a_n^{(i)})$ with $a_n^{(i)} \leq a_n^{(j)}$ for all $i < j$ and $n \in \mathbb{N}$, let us now consider an arbitrary sequence $(a'_n) \in \mathfrak{a}'$. Since $\mathfrak{a}' > \mathfrak{a}_i$ for all $i \in \mathbb{N}$, we have that for any fixed $i \in \mathbb{N}$, the sequence of ratios $a'_n/a_n^{(i)}$ is unbounded above. We may therefore define a sequence of indices (m_i) as follows: let $m_1 = 1$, and let m_{i+1} be the smallest natural number strictly greater than m_i such that $a'_m/a_m^{(i)} \geq i$.

We are now ready to use a "diagonalization" technique to define a sequence (b_n) with an intermediate growth order. Define the sequence as follows:

$$b_n = \begin{cases} a_n^{(i)} & \text{if } n = m_i,\ i \in \mathbb{N} \\ a'_n & \text{else} \end{cases}$$

We can show that if $\mathfrak{b} = [(b_n)]$, then $\mathfrak{b} < \mathfrak{a}'$ while $\mathfrak{b} > \mathfrak{a}_i$ for all $i \in \mathbb{N}$. First of all: for any fixed $i \in \mathbb{N}$, we have for all $n > m_i$ that $b_n \geq a_n^{(i+1)}$ (which can be seen easily by considering the two cases in the definition of b_n) and therefore $\mathfrak{b} \geq \mathfrak{a}_{i+1} > \mathfrak{a}_i$. Secondly, we may deduce that $\mathfrak{b} < \mathfrak{a}'$ by considering the ratio b_n/a'_n. For any $n \in \mathbb{N}$, we either have that $b_n/a'_n = 1$ (when $n \neq m_i$ for any $i \in \mathbb{N}$) or $b_n/a'_n \leq 1/n$ (when $n = m_i$, because m_i is defined such that $a_{m_i}^{(i)}/a'_{m_i} \leq 1/m_i$). Thus, we have that the sequence (b_n/a'_n) is bounded above by 1 but comes arbitrarily close to 0, meaning that $\mathfrak{b}/\mathfrak{a}' < \mathbf{1}$ and therefore $\mathfrak{b} < \mathfrak{a}'$. Thus, we have constructed $\mathfrak{b}$ such that

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{b} < \mathfrak{a}'$$

as claimed. □

Although we have just proven that no strictly increasing sequence of growth orders in $\mathcal{S}(\mathbb{R}^+)$ has a *least upper bound*, it is in fact true that every increasing sequence of growth orders has *some* upper bound.

Proposition 20. *For any chain of growth orders*

$$\mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \mathfrak{a}_3 \leq \cdots$$

there exists a growth order $\mathfrak{a}'$ *such that* $\mathfrak{a}' \geq \mathfrak{a}_i$ *for all* $i \in \mathbb{N}$.

Proof. We can complete this proof using a diagonalization argument. Let us choose one sequence from each growth order $(a_n^{(i)}) \in \mathfrak{a}_i$ (making use of the Axiom of Choice). Then we may define a sequence (a'_n) as follows:

$$a'_n = \sup_{1 \leq i \leq n} a_n^{(i)}$$

so that $a'_n \geq a_n^{(i)}$ for all $n \geq i$, for all $i \in \mathbb{N}$. This means that if $\mathfrak{a}' = [(a'_n)]$, we have that $\mathfrak{a}' \geq \mathfrak{a}_i$ for all $i \in \mathbb{N}$, as desired. □

We have seen that the ordering on $\mathcal{S}(\mathbb{R}^+)$ differs from the ordering on, say, $\mathbb{R}^+$ in several key ways: for one, bounded sequences in $\mathbb{R}^+$ always have least upper bounds, which is not true in this poset; on the other hand, not all sequences in $\mathbb{R}^+$ have a upper bound *at all*, but in this poset all sequences are bounded.

4. Summation operators

4.1. Partial summation

In my mind, one of the principal motivations for setting up this whole theory was to address the following question: given a sequence of known growth order, how can we determine the growth order of its sequence of partial sums? That is, given a sequence (a_n), are there any general rules or principles allowing us to deduce the growth order of

$$\sum_{i=1}^{n} a_i \sim ?$$

At first, I was just as interested in finding "quick and dirty" tricks for calculating asymptotic formulas for sums that appeared, for instance, in computational complexity considerations for algorithms in computer science. However, I quickly found out that the situation was much more interesting than I'd expected.

Let's begin by defining this as an operation on sequences, and showing that it is well-defined as an operation on growth orders.

Definition 9. *Given a sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$, define its sequence of* ***partial sums****, denoted $\Sigma\alpha$, as the sequence*

$$\left(\sum_{i=1}^{n} a_i\right)$$

.

Proposition 21. *If $\alpha \sim \alpha'$, then $\Sigma\alpha \sim \Sigma\alpha'$.*

Proof. Suppose that $\alpha \sim \alpha'$, so that there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a_n' \leq C_2 a_n$$

for all $n \in \mathbb{N}$. It follows that

$$C_1 \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a_n' \leq C_2 \sum_{i=1}^{n} a_i$$

so that we have $\Sigma\alpha \sim \Sigma\alpha'$ by definition. □

Therefore, the following definition is justified:

Definition 10. *Given a growth order $\mathfrak{a} = [\alpha]$, define its* ***partial sum*** *to be the growth order $\Sigma\mathfrak{a} = [\Sigma\alpha]$.*

Here are some elementary properties of this new operation:

Proposition 22. *The following facts hold for arbitrary growth orders $\mathfrak{a}, \mathfrak{b}$:*
- $\Sigma\mathfrak{a} \geq \mathbf{1}$
- $\Sigma\mathfrak{a} \geq \mathfrak{a}$
- $\mathfrak{a} \leq \mathfrak{b} \implies \Sigma\mathfrak{a} \leq \Sigma\mathfrak{b}$
- $\Sigma(\mathfrak{a} + \mathfrak{b}) = \Sigma\mathfrak{a} + \Sigma\mathfrak{b}$

Proof. Trivial. □

Notice that the inequality $\mathfrak{a} \neq \mathfrak{b}$ does not imply $\Sigma\mathfrak{a} \neq \Sigma\mathfrak{b}$ in general - that is, Σ is not injective as a function on growth orders. It is not hard to come up with an example of two incomparable growth orders $\mathfrak{a} \perp \mathfrak{b}$ such that $\Sigma\mathfrak{a} = \Sigma\mathfrak{b}$. For example, consider the sequences $\mathfrak{n}$ and $\mathfrak{u}$, which are unequal despite the fact that their partial sums have the same growth order $\Sigma\mathfrak{n} = \Sigma\mathfrak{u} = \mathfrak{n}^2$.

At this point, we might wonder whether this problem only arises when $\mathfrak{a} \perp \mathfrak{b}$. That is, if $\mathfrak{a} \neq \mathfrak{b}$ *and* $\mathfrak{a}, \mathfrak{b}$ are comparable, then perhaps from this we can deduce that $\Sigma\mathfrak{a} \neq \Sigma\mathfrak{b}$? Alas, this also fails to be true. As a counterexample, consider $\mathfrak{a} = \mathbf{1}$ and $\mathfrak{b}$ defined as the growth order of the sequence $\beta = (b_n)$ defined piecewise as follows:

$$b_n = \begin{cases} k & \text{if } n = 2^k \\ 1 & \text{else} \end{cases}$$

In this case, we have $\mathfrak{a} < \mathfrak{b}$ because $\mathfrak{b}$ is bounded below by 1 yet is unbounded, and the entries of $\Sigma\beta$ are $n + O(\log^2 n)$, meaning that $\Sigma\mathfrak{a} = \Sigma\mathfrak{b}$.

With a bit of effort, we may prove that, by taking partial sums, we will never convert a moderate growth sequence into a non-moderate sequence. That is, the partial sum operation preserves moderate growth.

Proposition 23. *If $\mathfrak{a}$ is moderate, then $\Sigma\mathfrak{a}$ is moderate.*

Proof. Let $k \in \mathbb{N}$ be given. By the moderateness of $\mathfrak{a}$, for any $\alpha = (a_n) \in \mathfrak{a}$, there exist constants C_1, C_2 such that for all m with $n \leq m \leq kn$, we have

$$C_1 a_n \leq a_m \leq C_2 a_n$$

Now let us fix some $m, n \in \mathbb{N}$ with $n \leq m \leq kn$, and consider the sum

$$\sum_{i=1}^{m} a_i$$

Because the a_i are positive and $m \geq n$, we clearly have that

$$\sum_{i=1}^{m} a_i \geq \sum_{i=1}^{n} a_i$$

On the other hand, we have that

$$\begin{aligned}\sum_{i=1}^{m} a_i &\leq \sum_{i=1}^{kn} a_i \\ &= \sum_{i=1}^{n} a_i + \sum_{j=0}^{n-1} \sum_{i=1}^{k} a_{jk+i} \\ &\leq \sum_{i=1}^{n} a_i + \sum_{j=0}^{n-1} \sum_{i=1}^{k} C_2 a_{jk+1}\end{aligned}$$

Because $jk + 1 \leq jk + i \leq k(jk + 1)$ for all $k \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$, and $i \in \{1, \cdots, k\}$. We may further simplify this upper bound as follows:

$$\begin{aligned}\sum_{i=1}^{m} a_i &\leq \sum_{i=1}^{n} a_i + \sum_{j=0}^{n-1} kC_2 a_{jk+1} \\ &\leq \sum_{i=1}^{n} a_i + \sum_{j=0}^{n-1} kC_2^2 a_{j+1} \\ &= (1 + kC_2^2) \sum_{i=1}^{n} a_i\end{aligned}$$

since $j + 1 \leq jk + 1 \leq k(j + 1)$ for all $j \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Thus, the sum can be bounded both above and below as follows:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{m} a_i \leq (1 + kC_2^2) \sum_{i=1}^{n} a_i$$

which proves that $\Sigma\alpha$ has moderate growth by definition, and that $\mathfrak{a}$ is moderate as claimed. $\square$

Proposition 24. *If $\mathfrak{a}$ is moderate, then $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$.*

Proof. Let $(a_n) \in \mathfrak{a}$ be moderate. From the previous proposition, we have that

$$\left(\sum_{i=1}^{n} a_i\right) \sim \left(\sum_{i=1}^{2n} a_i\right)$$

and we may split up this sum as follows:

$$\sum_{i=1}^{2n} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{2n} a_i$$

From 7, we have that this is $\sim \Sigma(a_n) + (na_n)$, which grows at least as fast as (na_n), with growth order $\mathfrak{n}\mathfrak{a}$. Thus, we have that $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$ as claimed. □

4.2. Monotone growth orders

Definition 11. *We say that a growth order $\mathfrak{a}$ is **monotone** if it contains some monotone sequence $\alpha = (a_n) \in \mathfrak{a}$.*

Proposition 25. *If $\mathfrak{a}$ is monotone, then it is comparable to $\mathbf{1}$.*

Proof. Suppose that $\alpha = (a_n) \in \mathfrak{a}$ is a monotone sequence. If (a_n) is monotone increasing, then it is bounded below by a_1, and therefore $\mathfrak{a} \geq \mathbf{1}$. If it is monotone decreasing, then it is bounded above by a_1, and we have that $\mathfrak{a} \leq \mathbf{1}$. □

A natural question arises from this fact: if all monotone growth orders are comparable to $\mathbf{1}$, might it be the case that all monotone growth orders are comparable *amongst themselves*? Unfortunately, this is not the case. As a counterexample, consider the growth orders $\mathfrak{a} = \mathfrak{n}^{1/2}$ and $\mathfrak{b}$ defined as the growth order of the sequence $\beta = (b_n)$ defined by $b_1 = 1$ and

$$b_n = 2^{3^{\lfloor \log_3 \log_2 n \rfloor}} = \exp_2 \exp_3 \lfloor \log_3 \log_2 n \rfloor$$

for all $n \geq 2$. Clearly $\mathfrak{a}$ is monotone, and $\mathfrak{b}$ is monotone because each of the functions $\exp_2, \exp_3, \lfloor \cdot \rfloor, \log_2, \log_3$ is monotone and because $b_1 = 1 < b_2 = 2$. Notice that when $n = 2^{3^k}$ for some $k \in \mathbb{N}$, we have that $b_n = n$, whereas when $n = 2^{3^k - 1}$, we have that $b_n = n^{1/3}$. Hence, if $\alpha \in \mathfrak{a}$, then α/β is unbounded on the subsequence $n = 2^{3^k - 1}$, and β/α is unbounded on the subsequence $n = 2^{3^k}$. This means that $\mathfrak{a} \perp \mathfrak{b}$ despite the fact that $\mathfrak{a}, \mathfrak{b}$ are both monotone! Apparently, monotonicity comes with some *limited* guarantees of comparability, but not all of the guarantees that we might hope for.

Monotonicity is a helpful property of sequences because of the above fact, namely that it ensures comparability to the constant growth order. This is something that our previous "niceness" condition, namely moderateness, fails to accomplish. For examples, consider the sequence (a_n) defined by

$$a_n = n^{\sin\log\log n}$$

for $n \geq 3$. Clearly this sequence is incomparable to $\mathbf{1}$, since it has subsequences tending to 0 and to ∞. It is, however, moderate. To see why notice that, whenever $n \leq m \leq kn$, we have

$$|\log\log m - \log\log n| \leq |\log\log kn - \log\log n| \sim \frac{\log k}{\log n}$$

and, since the sine function is Lipschitz continuous, this means that

$$|\log_n a_m - \log_n a_n| \leq \frac{K}{\log n}$$

for some constant $K \in \mathbb{R}^+$ depending only on k. This means that a_m/a_n is bounded above by e^K and below by e^{-K}, proving that (a_n) is moderate by definition. This shows that moderateness is not sufficient to guarantee comparability to $\mathbf{1}$, which, hopefully, allows us to appreciate why monotonicity is useful as a secondary "niceness" condition.

Question 2 What exactly is the growth order of the sum

$$\sum_{k=2}^{n} k^{\sin\log\log k} = \Theta(?)$$

In an earlier section, we proved a fundamental property of the partial ordering on $\mathcal{S}(\mathbb{R}^+)/\sim$, namely that no strictly increasing sequence of growth orders has a least upper bound. That is, given any sequence of growth orders $\mathfrak{a}_i$ with upper bound $\mathfrak{a}'$ such that

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{a}'$$

there always exists a growth order $\mathfrak{b}$ which is simultaneously greater than each of the $\mathfrak{a}_i$ and strictly less than $\mathfrak{b}$. Later in this section and future sections, we shall see that when comparing growth orders, it is often useful to check not only whether they are comparable, but also whether their *quotient is monotone*, which gives a great deal of additional useful information. For this reason, we will now prove a version of the previously mentioned property of the ordering on $\mathcal{S}(\mathbb{R}^+)/\sim$ that gives a stronger guarantee for increasing sequences of growth orders whose respective quotients are monotone.

Proposition 26. *For any strictly increasing sequence of growth orders*

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{a}'$$

such that each of the quotients $\mathfrak{a}_i/\mathfrak{a}_j$ and $\mathfrak{a}'/\mathfrak{a}_i$ is monotone, there exists a growth order $\mathfrak{b}$ such that $\mathfrak{b} > \mathfrak{a}_i$ for all $i \in \mathbb{N}$ and $\mathfrak{b} < \mathfrak{a}'$, and all of the quotients $\mathfrak{b}/\mathfrak{a}_i$ and $\mathfrak{b}/\mathfrak{a}'$ are monotone.

Proof. First of all, notice that it suffices to prove this claim for $\mathfrak{a}' = \mathbf{1}$. For if this seemingly weaker claim holds, suppose that we are given some growth orders $\mathfrak{a}_i$ and $\mathfrak{a}'$ with

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{a}'$$

and all pairwise quotients monotone. Consider the chain

$$\mathfrak{a}_1/\mathfrak{a}' < \mathfrak{a}_2/\mathfrak{a}' < \mathfrak{a}_3/\mathfrak{a}' < \cdots < \mathbf{1}$$

If the weaker claim were true, then we would have that there exists a growth order $\mathfrak{c}$ such that

$$\mathfrak{a}_1/\mathfrak{a}' < \mathfrak{a}_2/\mathfrak{a}' < \mathfrak{a}_3/\mathfrak{a}' < \cdots < \mathfrak{c} < \mathbf{1}$$

and such that $\mathfrak{c}$ and all of the quotients $\mathfrak{c}/(\mathfrak{a}_i/\mathfrak{a}') = \mathfrak{c}\mathfrak{a}'/\mathfrak{a}_i$ are monotone. But from this it would follow, by multiplying through by the growth order $\mathfrak{a}'$, that

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{c}\mathfrak{a}' < \mathfrak{a}'$$

with the ratio $\mathfrak{a}'/(\mathfrak{c}\mathfrak{a}') = \mathfrak{c}^{-1}$ monotone (because $\mathfrak{c}$ is monotone) and the ratios $\mathfrak{c}\mathfrak{a}'/\mathfrak{a}_i$ monotone as explained above. Thus, if we prove the weakened claim for the case of an upper bound of $\mathbf{1}$, the more general version will follow immediately.

Suppose, then, that we have some strictly ascending sequence of growth orders

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathbf{1}$$

and let $(a_n^{(i)}) \in \mathfrak{a}_i$ be a sequence of monotone decreasing sequences from these growth orders such that each of the sequences $(a_n^{(i)}/a_n^{(j)})$ is monotone, selected using the Axiom of Choice. We will now recursively define another sequence of sequences, each of which is a scalar multiple of one of the $(a_n^{(i)})$. Let us call these sequences $(b_n^{(i)})$, and define them as follows. Start by letting $(b_n^{(1)}) = (a_n^{(1)})$. Then define an increasing sequence of indices (m_i) by letting $m_1 = 1$ and m_i be the smallest natural number greater than m_{i-1} such that $b_n^{(i)} \le 1/i$ for $i > 1$. Additionally, define a sequence of constants (C_i) by letting $C_1 = 1$ and $C_i = a_{m_{i-1}}^{(i)}/b_{m_{i-1}}^{(i-1)}$ for all $i > 1$. Finally, define the sequence $(b_n^{(i)}) = (a_n^{(i)}/C_i)$ for each $i > 1$. We my now define the sequence (b_n) in terms of the sequences $(b_n^{(i)})$ as follows:

$$b_n = \begin{cases} b_1^{(1)} & \text{if } n = m_1 = 1 \\ b_n^{(i)} & \text{if } m_{i-1} < n \le m_i,\ i > 1 \end{cases}$$

We will now show that (b_n) is a monotone decreasing sequence tending to zero, while each of the sequences $(b_n/a_n^{(i)})$ is monotone increasing for all $n \ge m_i$. Notice that when $n = m_i$, we have that $b_n^{(i)} = b_n^{(i+1)}$. Therefore, since each of the sequences $(b_n^{(i)})$ is monotone decreasing, and (b_n) is defined piecewise as these sequences on successive intervals such that the sequences have the same value at the points where (b_n) "switches" from $(b_n^{(i)})$ to $(b_n^{(i+1)})$, we have that (b_n) is also monotone decreasing. Further, we have that $b_{m_i} \le 1/i$ by the definition of the indices (m_i), meaning that (b_n) also tends to zero. Hence, if $\mathfrak{b}$ is the growth order of (b_n), we have that $\mathfrak{b} < \mathbf{1}$ with a monotone ratio.

Now, notice that because each $(b_n^{(i)})$ is defined as a scalar multiple of the corresponding $(a_n^{(i)})$, and the ratios $(a_n^{(i+1)}/a_n^{(i)})$ are monotone, we have that the ratios $(b_n^{(i+1)}/b_n^{(i)})$ are also monotone. Now, if $n \geq m_i$ for some fixed $i \in \mathbb{N}$, we can see that

$$\frac{b_{n+1}}{b_{n+1}^{(i)}} \geq \frac{b_n}{b_n^{(i)}}$$

because if $m_j \leq n \leq m_{j+1}$ for some $j \geq i$, this inequality becomes

$$\frac{b_{n+1}^{(j+1)}}{b_{n+1}^{(i)}} \geq \frac{b_n^{(j)}}{b_n^{(i)}}$$

which follows from the monotone increasing nature of the ratio sequences $(b_n^{(j)}/b_n^{(i)})$ for $j \geq i$. This means that $(b_n/b_n^{(i)})$ is *eventually monotone* for every fixed $i \in \mathbb{N}$, and since each $(b_n^{(i)})$ has growth order $\mathfrak{a}_i$, we have that $\mathfrak{b}/\mathfrak{a}_i$ is monotone increasing for each $i \in \mathbb{N}$.

We may therefore conclude that

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{b} < \mathfrak{a}'$$

such that each of the ratios $\mathfrak{b}/\mathfrak{a}_i$ and $\mathfrak{a}'/\mathfrak{b}$ is monotone, proving the claimed existence of such a growth order $\mathfrak{b}$. □

Now we will begin to explore the relationship between monotonicity of growth orders and the partial summation operator Σ.

Proposition 27. *If $\mathfrak{a} > \mathbf{1}$, then $\mathfrak{a}$ is monotone if and only if $\mathfrak{a} = \Sigma\mathfrak{b}$ for some other growth order $\mathfrak{b}$.*

Proof. First, suppose $\mathfrak{a}$ is monotone, and that $\alpha = (a_n) \in \mathfrak{a}$ is a monotone sequence. It must be monotone increasing, since $\mathfrak{a} > \mathbf{1}$. If we define the sequence (b_n) by letting $b_1 = a_1 + \frac{1}{2}$ and

$$b_{n+1} = a_{n+1} - a_n + \frac{1}{2^n}$$

then we have that

$$\sum_{i=1}^{n} b_n = a_n + 1 - \frac{1}{2^{n+1}}$$

for all $n \in \mathbb{N}$, and since $\mathfrak{a} > \mathbf{1}$, we have that the constant term is negligible and $\Sigma\mathfrak{b} = \mathfrak{a}$ as desired.

The converse is trivial, for if $\alpha = \Sigma\beta$ for some sequences $\alpha = (a_n) \in \mathfrak{a}$ and $\beta = (b_n) \in \mathfrak{b}$, then α is monotone because $a_{n+1} - a_n = b_n$ is strictly positive for all $n \in \mathbb{N}$. □

Question 3 If $\mathfrak{a} > \mathbf{1}$ is both moderate and monotone, is it guaranteed that there exists a *moderate* growth order $\mathfrak{b}$ such that $\mathfrak{a} = \Sigma\mathfrak{b}$?

Proposition 28. *A growth order $\mathfrak{a}$ is monotone if and only if $\mathfrak{a} = \mathbf{1}$ or $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$ for some growth order $\mathfrak{b}$.*

Proof. We know from the previous proposition that the growth orders of the form $\Sigma\mathfrak{b}$ are always monotone, meaning that those in the form $(\Sigma\mathfrak{b})^{-1}$ are monotone as well. Thus, the "if" direction is trivial.

On the other hand, if $\mathfrak{a}$ is monotone, we know from a previous proposition that it is comparable to $\mathbf{1}$. If $\mathfrak{a} \neq \mathbf{1}$, then either $\mathfrak{a} > \mathbf{1}$, in which case there exists $\mathfrak{b}$ such that $\mathfrak{a} = \Sigma\mathfrak{b}$, or $\mathfrak{a} < \mathbf{1}$, in which case $\mathfrak{a}^{-1} > \mathbf{1}$ and there exists $\mathfrak{b}$ such that $\mathfrak{a}^{-1} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$. Thus we have proven the "only if" direction. □

Proposition 29. *If $\mathfrak{a}/\mathfrak{b} > 1$ is monotone, then $\Sigma\mathfrak{a}/\Sigma\mathfrak{b}$ is monotone.*

Proof. First notice that, if $\mathfrak{a}/\mathfrak{b}$ is monotone, then we can choose $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ such that a_n/b_n is monotone. Specifically, if we choose an arbitrary monotone sequence $(r_n) \in \mathfrak{a}/\mathfrak{b}$ and and arbitrary sequence $(b_n) \in \mathfrak{b}$, then defining $(a_n) \in \mathfrak{a}$ by the equation $a_n = r_n b_n$ accomplishes this, ensuring that $(a_n/b_n) = (r_n b_n/b_n) = (r_n)$ is monotone.

Now we will make use of the inequality

$$\frac{x}{y} \leq \frac{x+x'}{y+y'} \leq \frac{x'}{y'}$$

which applies to all $x, x', y, y' \in \mathbb{R}^+$ with $x'/y' \geq x/y$. Consider the following inequality:

$$\frac{a_{n+1}}{b_{n+1}} \geq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

clearly this is true for $n = 1$ by the monotonicity of a_n/b_n, so $n = 1$ will serve as our base case. Now suppose that this inequality holds for some $n \in \mathbb{N}$, and for all preceding values. By the inequality mentioned at the beginning of the proof and the monotonicity of a_n/b_n, we have that

$$\frac{a_{n+2}}{b_{n+2}} \geq \frac{a_{n+1}}{b_{n+1}} \geq \frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} \geq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

This inequality establishes both of the following inequalities:

$$\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} \geq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

$$\frac{a_{n+2}}{b_{n+2}} \geq \frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i}$$

the former of which proves that the sequence $\Sigma\alpha/\Sigma\beta$ is monotonic up to index $n+1$, and the latter of which extends our original assumption from case n to case $n+1$, allowing us to inductively prove our claim for all $n \in \mathbb{N}$. $\square$

Proposition 30. *If* $\mathfrak{a}/\mathfrak{b} > \mathbf{1}$ *is monotone and* $\Sigma\mathfrak{a} > \mathbf{1}$*, then* $\Sigma\mathfrak{a}/\Sigma\mathfrak{b} > \mathbf{1}$.

Proof. We may use the same construction as in the previous proof to choose $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$ such that α/β is monotone. Letting $R \in \mathbb{R}^+$ be arbitrary, we will show that $\Sigma\alpha/\Sigma\beta$ eventually exceeds R, and is therefore unbounded.

Because α/β is monotone and > 1, there exists $N \in \mathbb{N}$ such that $a_n/b_n > 2R$ for all $n \geq N$. Furthermore, since $\Sigma\mathfrak{a} > \mathbf{1}$, we have that $\Sigma\mathfrak{a}$ is unbounded, and there therefore exists $M \in \mathbb{N}$ such that

$$\sum_{i=N}^{M} a_i \geq 2R \sum_{i=1}^{N-1} b_i$$

Then we have the following inequality:

$$\frac{\sum_{i=1}^{K} a_i}{\sum_{i=1}^{K} b_i} = \frac{\sum_{i=1}^{N-1} a_i + \sum_{i=N}^{K} a_i}{\sum_{i=1}^{N-1} b_i + \sum_{i=N}^{K} b_i} > \frac{\sum_{i=N}^{K} a_i}{\sum_{i=1}^{N-1} b_i + \sum_{i=N}^{K} b_i}$$

Now notice that the numerator of this ratio is greater than or equal to R times the denominator, since it is greater than or equal to $2R$ times each of the sums in the denominator. Thus, we have that

$$\frac{\sum_{i=1}^{K} a_i}{\sum_{i=1}^{K} b_i} > R$$

and, since $\Sigma\alpha/\Sigma\beta$ is monotone by the previous proposition, we have that all elements of $\Sigma\alpha/\Sigma\beta$ with $n \geq K$ exceed R. Hence, $\Sigma\alpha/\Sigma\beta$ is unbounded above (since R was arbitrary) and therefore $> \mathbf{1}$, since it is monotone. $\square$

Recall that, in the first section of this chapter, we found a troublesome counterexample in which Σ failed to preserve strict inequality of sequences. Using the two propositions above, we are now prepared to "salvage" this idea by providing sufficient conditions for $\mathfrak{a} < \mathfrak{b}$ to imply $\Sigma\mathfrak{a} < \Sigma\mathfrak{b}$.

Proposition 31. *If* $\mathfrak{a}/\mathfrak{b}$ *is monotone and* $\Sigma\mathfrak{b} > \mathbf{1}$*, then* $\mathfrak{a} < \mathfrak{b} \implies \Sigma\mathfrak{a} < \Sigma\mathfrak{b}$.

Proof. This follows easily from the above two propositions. If $\mathfrak{a} < \mathfrak{b}$, then $\mathfrak{b}/\mathfrak{a}$ is monotone (since $\mathfrak{a}/\mathfrak{b}$ is monotone by hypothesis) and it is also $> \mathbf{1}$. By the previous two propositions, since $\Sigma\mathfrak{b} > \mathbf{1}$, we have that $\Sigma\mathfrak{b}/\Sigma\mathfrak{a} > \mathbf{1}$ and therefore $\Sigma\mathfrak{a} < \Sigma\mathfrak{b}$ as claimed. $\square$

Question 4 Is Σ injective on moderate growth orders? Do there exist two moderate growth orders $\mathfrak{a} \neq \mathfrak{b}$ such that $\Sigma\mathfrak{a} = \Sigma\mathfrak{b}$?

4.3. The partial sum ratio

There are several asymptotic formulas for partial sums that are familiar from calculus. For instance:

$$\begin{aligned}
&\Sigma\mathfrak{n}^p = \mathfrak{n}^{p+1}\\
&\Sigma\mathfrak{n}^p\mathfrak{l}^q = \mathfrak{n}^{p+1}\mathfrak{l}^q\\
&\Sigma\mathfrak{n}^{-1} = \mathfrak{l}\\
&\Sigma\mathfrak{n}^{-1}\mathfrak{l}^p = \mathfrak{l}^{p+1}\\
&\Sigma(\mathfrak{n}\mathfrak{l})^{-1} = [(\log\log n)]
\end{aligned}$$

for $p \in (-1, \infty)$ and $q \in \mathbb{R}$. If we look for patterns or tricks that might allow us to quickly calculate the asymptotics of a sequence of partial sums, the first thing that pops out is that, for a broad class of growth orders, taking partial sums amounts to just multiplying the original growth order by $\mathfrak{n}$. This is the case for the first two classes of growth orders: Σ sends $\mathfrak{n}^p$ to $\mathfrak{n} \cdot \mathfrak{n}^p = \mathfrak{n}^{p+1}$ and sends $\mathfrak{n}^p\mathfrak{l}^q$ to $\mathfrak{n} \cdot \mathfrak{n}^p\mathfrak{l}^q = \mathfrak{n}^{p+1}\mathfrak{l}^q$. However, for the growth order $\mathfrak{n}^{-1}$, taking partial sums increases the original growth order by a factor of $\mathfrak{n}\mathfrak{l}$, rather than $\mathfrak{n}$. And for the growth order $(\mathfrak{n}\mathfrak{l})^{-1}$, taking partial sums increases it by a factor of $[(n\log n\log\log n)]$.

While these examples don't suggest any obvious catch-all technique for determining the growth order of $\Sigma\mathfrak{a}$ in general (despite some noticeable patterns for special cases like $\mathfrak{n}^p\mathfrak{l}^q$), they do hint that it may be informative to study the *factor by which a growth order increases* when we take its partial sums. That is, we should take a closer look not just at the Σ function, but at the function which sends $\mathfrak{a} \to \mathfrak{a}/\Sigma\mathfrak{a}$. Let us denote this **partial sum ratio** $\mathfrak{a}/\Sigma\mathfrak{a}$ using the notation $\mathrm{P}\mathfrak{a}$. From our observations above, we know, for instance, that $\mathrm{P}\mathfrak{a} = \mathfrak{n}^{-1}$ when $\mathfrak{a}$ takes the form $\mathfrak{n}^p\mathfrak{l}^q$ with $p > -1$, and that $\mathrm{P}\mathfrak{a} = (\mathfrak{n}\mathfrak{l})^{-1}$ when $\mathfrak{a}$ takes the form $\mathfrak{n}^{-1}\mathfrak{l}^p$ with $p > -1$. It seems that P is constant for large intervals of growth orders.

Nevertheless, the above proposition shows that for sequences with monotone ratios, the partial sum ratio P preserves their order. Clearly this transformation is not necessarily *strictly* order-preserving on such sequences, because, as we saw earlier, it maps many different growth orders to the same ratio.

Proposition 32. *If* $\mathfrak{a} \leq \mathfrak{b}$ *and* $\mathfrak{a}/\mathfrak{b}$ *is monotone, then* $\mathrm{P}\mathfrak{a} \leq \mathrm{P}\mathfrak{b}$.

Proof. Suppose $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a}/\mathfrak{b}$ is monotone, so that there exists a monotone decreasing sequence $(c_n) \in \mathfrak{a}/\mathfrak{b}$, and we may choose $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ such that $a_n/b_n = c_n$. (Let (b_n) be an arbitrary element of $\mathfrak{b}$ and let (a_n) be defined by $a_n = b_nc_n$.) Then we have that

$$c_n\sum_{k=1}^n b_k = \sum_{k=1}^n c_nb_k \leq \sum_{k=1}^n c_kb_k = \sum_{k=1}^n a_k$$

so we have that $(\mathfrak{a}/\mathfrak{b})\Sigma\mathfrak{b} \leq \Sigma\mathfrak{a}$, which is equivalent to $\mathrm{P}\mathfrak{a} \leq \mathrm{P}\mathfrak{b}$. □

The implication $\mathfrak{a} \leq \mathfrak{b} \implies \mathrm{P}\mathfrak{a} \leq \mathrm{P}\mathfrak{b}$ seems to suggest that P is a monotone increasing function on growth orders. However, the additional stipulation that $\mathfrak{a}/\mathfrak{b}$ be monotone is essential. Consider the two growth orders $\mathfrak{n}^{-1/2}$ and $\mathfrak{u}^{1/3}$, which satisfy $\mathfrak{n}^{-1/2} < \mathfrak{u}^{1/3}$. We also have $\Sigma\mathfrak{n}^{1/2} = \mathfrak{n}^{3/2}$ and $\Sigma\mathfrak{u}^{1/3} = \mathfrak{n}^{4/3}$, so that $\mathrm{P}\mathfrak{n}^{-1/2} = \mathfrak{n}^{-1}$ and $\mathrm{P}\mathfrak{u}^{1/3} = \mathfrak{u}^{1/3}\mathfrak{n}^{-4/3}$, so that $\mathrm{P}\mathfrak{u}^{1/3} < \mathrm{P}\mathfrak{n}^{-1/2}$. Thus, not only does P fail to be monotone increasing in general, but it actually *reverses* the order of some growth orders, such as $\mathfrak{n}^{-1/2} < \mathfrak{u}^{1/3}$ with $\mathrm{P}\mathfrak{u}^{1/3} < \mathrm{P}\mathfrak{n}^{-1/2}$.

Proposition 33. *For any moderate growth order $\mathfrak{a}$ with $(a_n) \in \mathfrak{a}$, we have that $\Sigma\mathrm{P}\mathfrak{a}$ is the growth order of the sequence (b_n) defined by*

$$b_n = \log\left(1 + \sum_{i=1}^{n} a_i\right)$$

Proof. From the above definition of b_n, we have

$$b_{n+1} - b_n = \log\left(1 + \sum_{i=1}^{n+1} a_i\right) - \log\left(1 + \sum_{i=1}^{n} a_i\right) = \log\left(1 + \frac{a_{n+1}}{\sum_{i=1}^{n} a_i}\right)$$

and by 24 we have that the ratio inside of the logarithm on the RHS decays at least as fast as $\mathfrak{n}^{-1}$. We know from analysis that $\log(1+h)$ is $\Theta(h)$ as $h \to 0$, so we have that

$$b_{n+1} - b_n \sim \frac{a_{n+1}}{\sum_{i=1}^{n} a_i}$$

The sequence with terms given by the RHS of this asymptotic equivalence has growth order $\mathfrak{a}/\Sigma\mathfrak{a}$ (because $\mathfrak{a}$ is moderate). The partial sums of the RHS yield the original sequence (b_n), so we have that $\Sigma(\mathfrak{a}/\Sigma\mathfrak{a})$, or $\Sigma\mathrm{P}\mathfrak{a}$, is the growth order of (b_n), as claimed. □

We can also prove the following similar formula, which can be thought of as an analogue of the *power rule* from calculus:

Proposition 34. *For any moderate growth order $\mathfrak{a}$ with $(a_n) \in \mathfrak{a}$, and for any $p > -1$, we have that $\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p) = (\Sigma\mathfrak{a})^{p+1}$, and for any $p < -1$, we have $\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p) = \mathbf{1}$.*

Proof. Let $p > -1$, and define a sequence (b_n) as follows:

$$b_n = \left(\sum_{i=1}^{n} a_i\right)^{p+1}$$

Using the same technique as the previous proof, we have that

$$b_{n+1} - b_n = \left(a_{n+1} + \sum_{i=1}^{n} a_i\right)^{p+1} - \left(\sum_{i=1}^{n} a_i\right)^{p+1} = \left(\sum_{i=1}^{n} a_i\right)^{p+1}\left(\left(1 + \frac{a_{n+1}}{\sum_{i=1}^{n} a_i}\right)^{p+1} - 1\right)$$

Now recall from calculus that $(1+h)^{p+1}-1$ is $\Theta(h)$ as $h\to 0$, meaning that

$$b_{n+1}-b_n \sim \Big(\sum_{i=1}^n a_i\Big)^{p+1}\cdot\frac{a_{n+1}}{\sum_{i=1}^n a_i} = a_{n+1}\Big(\sum_{i=1}^n a_i\Big)^p$$

which has a growth order of $\mathfrak{a}(\Sigma\mathfrak{a})^p$, since $a_{n+1}\sim a_n$ (because $\mathfrak{a}$ is moderate). The partial sums of the LHS yield the sequence $(b_{n+1}-b_1)\sim(b_n)$, meaning that $\mathfrak{b}=(\Sigma\mathfrak{a})^{p+1}=\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p)$, as claimed.

Suppose, on the other hand, that $p<-1$. Because $\mathfrak{a}$ is moderate, we have that $\Sigma\mathfrak{a}\geq\mathfrak{n}\mathfrak{a}$, and therefore $\mathfrak{a}(\Sigma\mathfrak{a})^p\leq\mathfrak{n}^p$. Since $\Sigma\mathfrak{n}^p=\mathbf{1}$, we have that $\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p)=\mathbf{1}$ as well. □

Proposition 35. *If $\alpha=(a_n)\in\mathfrak{a}$ is a sequence tending to zero, and the sequence $\beta=e^{\Sigma\alpha}$ has growth order $\mathfrak{b}$, then $\Sigma(\mathfrak{a}\mathfrak{b})=\mathfrak{b}$.*

Proof. The equation $\beta=(b_n)=e^{\Sigma\alpha}$ simply means that

$$b_n=e^{\sum_{k=1}^n a_k}$$

Now, notice that

$$b_n-b_{n-1}=e^{\sum_{k=1}^n a_k}-e^{\sum_{k=1}^{n-1} a_k}=e^{\sum_{k=1}^n a_k}(1-e^{-a_n})$$

Because $1-e^{-h}=\Theta(h)$ as $h\to 0$, and (a_n) is a sequence tending to zero, we have that the sequence $(1-e^{-a_n})$ has the same growth order $\mathfrak{a}$ as (a_n), and therefore that the difference b_n-b_{n-1} has growth order $\mathfrak{a}\mathfrak{b}$. However, the partial sums of the sequence b_n-b_{n-1} are simply equal to b_n plus $O(1)$, and the $O(1)$ term can be neglected, since we already know that (b_n) is $\Omega(1)$ (since e^{a_n} is bounded below by 1 for $a_n>0$). Hence, we have that $\Sigma(\mathfrak{a}\mathfrak{b})=\mathfrak{b}$ as claimed. □

4.4. Convolution

Definition 12. *Given sequences $\alpha,\beta\in\mathcal{S}(\mathbb{R}^+)$, their **convolution** $\alpha*\beta$ is defined as the sequence (c_n) with entries given by*

$$c_n=\sum_{k=1}^n a_k b_{n-k+1}$$

Proposition 36. *If $\alpha,\alpha',\beta\in\mathcal{S}(\mathbb{R}^+)$ and $\alpha\sim\alpha'$, then $\alpha*\beta\sim\alpha'*\beta$.*

Proof. Suppose that $\alpha \sim \alpha'$ so that there exist constants C_1, C_2 such that

$$C_1 a_n' \leq a_n \leq C_2 a_n'$$

for all $n \in \mathbb{N}$. This implies that

$$C_1 \sum_{k=1}^{n} a_k' b_{n-k+1} \leq \sum_{k=1}^{n} a_k b_{n-k+1} \leq C_2 \sum_{k=1}^{n} a_k' b_{n-k+1}$$

and therefore $\alpha * \beta \sim \alpha' * \beta$ by the definition of convolution. □

Following the same pattern as before, this allows us to extend sequence convolution to a well-define operation on growth orders.

Definition 13. *If $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}(\mathbb{R}^+)/\sim$ are growth orders, define their* ***convolution*** $\mathfrak{a} * \mathfrak{b}$ *as the growth order of the sequence $\alpha * \beta$, where $\alpha \in \mathfrak{a}, \beta \in \mathfrak{b}$ are arbitrary.*

Here are some basic properties of convolution:

Proposition 37. *The following identities hold for all growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$:*

- $\mathfrak{a} * \mathfrak{b} = \mathfrak{b} * \mathfrak{a}$
- $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) = (\mathfrak{a} * \mathfrak{b}) * \mathfrak{c}$
- $\mathfrak{a} * (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} * \mathfrak{b}) + (\mathfrak{a} * \mathfrak{c})$
- $\mathfrak{a} * \mathbf{1} = \Sigma\mathfrak{a}$

Proof. We will show that these identities hold for all growth orders by showing that they hold for any of their constituent sequences. Let $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$, $(c_n) \in \mathfrak{c}$ be arbitrary.

First of all, we have that

$$\sum_{k=1}^{n} a_k b_{n-k+1} = \sum_{k=1}^{n} a_{n-k+1} b_k$$

by reindexing the sum, meaning that we have $(a_n) * (b_n) = (b_n) * (a_n)$ and therefore $\mathfrak{a} * \mathfrak{b} = \mathfrak{b} * \mathfrak{a}$.

For the second identity, we will make use of the following reformulation of the definition of convolution:

$$(\alpha * \beta)_n = \sum_{i+j=n+1} a_i b_j$$

where the sum ranges over positive integers i, j. This means that

$$\begin{aligned}((\alpha * \beta) * \gamma)_n &= \sum_{i+j=n+1} \sum_{k+l=i+1} a_k b_l c_j \\ &= \sum_{k+l+j=n+2} a_k b_l c_j \\ &= \sum_{k+i=n+1} \sum_{l+j=i+1} a_k b_l c_j \\ &= (\alpha * (\beta * \gamma))_n\end{aligned}$$

and therefore we have that $(\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = \mathfrak{a} * (\mathfrak{b} * \mathfrak{c})$.

For the next identity, we may simply use the fact that

$$\sum_{k=1}^{n} a_k (b_{n-k+1} + c_{n-k+1}) = \sum_{k=1}^{n} a_k b_{n-k+1} + \sum_{k=1}^{n} a_k c_{n-k+1}$$

which proves that $\mathfrak{a} * (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} * \mathfrak{b}) + (\mathfrak{a} * \mathfrak{c})$.

Finally, for the last identity, we may use the constant sequence $(1) \in \mathbf{1}$, which gives the following formula for the nth term of the convolution $(a_n) * (1)$:

$$\sum_{k=1}^{n} a_k \cdot 1$$

which is simply the nth partial sum of (a_n), meaning that $\mathfrak{a} * \mathbf{1} = \Sigma\mathfrak{a}$ as claimed. □

4.5. The convergence-divergence boundary

A natural question to ask while exploring convergent and divergent infinite series is the following: does there exist a growth order which exhibits the *slowest possible decay* for a sequence whose partial sums diverge? That is, does there exist a minimal growth order whose partial sums diverge?

Proposition 38. *For any growth order $\mathfrak{a}$ with $\Sigma\mathfrak{a} > \mathbf{1}$, there exists a growth order $\mathfrak{b} < \mathfrak{a}$ with $\Sigma\mathfrak{b} > \mathbf{1}$.*

Proof. Let us suppose that $\mathfrak{a} < \mathbf{1}$, or that $\mathfrak{a}$ is the growth order of a sequence tending to zero. For if this is not the case, we may simply let $\mathfrak{b} = \mathfrak{n}^{-1}$. If $(a_n) \in \mathfrak{a}$ tends to zero but $\Sigma\mathfrak{a} > \mathbf{1}$, then consider the sequence (b_n) defined by $b_1 = \sqrt{a_1}$ and

$$b_n = \sqrt{\sum_{i=1}^{n} a_i} - \sqrt{\sum_{i=1}^{n-1} a_i}$$

so that

$$\sum_{i=1}^{n} b_i = \sqrt{\sum_{i=1}^{n} a_i}$$

Now let $\mathfrak{b}$ be the growth order of this sequence. Clearly, since the partial sums of (a_n) tend to infinity, the partial sums of (b_n) also tend to infinity, since $x \mapsto \sqrt{x}$ is an unbounded strictly increasing function on $\mathbb{R}^+$. We also have that

$$\begin{aligned} b_n &= \left(\sqrt{\sum_{i=1}^{n-1} a_i}\right)\left(\sqrt{1 + \frac{a_n}{\sum_{i=1}^{n-1} a_i}} - 1\right) \\ &\sim \left(\sqrt{\sum_{i=1}^{n-1} a_i}\right) \cdot \frac{1}{2} \frac{a_n}{\sum_{i=1}^{n-1} a_i} \\ &= \frac{a_n}{2\sqrt{\sum_{i=1}^{n-1} a_i}} \end{aligned}$$

which has growth order $\mathfrak{a}/\sqrt{\Sigma\mathfrak{a}}$, which is strictly less than $\mathfrak{a}$, since $\Sigma\mathfrak{a} > \mathbf{1}$. Thus, we have found a growth order $\mathfrak{b}$ such that $\Sigma\mathfrak{b} > \mathbf{1}$ and $\mathfrak{b} < \mathfrak{a}$. □

This allows us to answer our question in the negative. There can be no "slowest diverging" infinite series, because for any growth order whose partial sums diverge, there exists a strictly lesser growth order whose partial sums also diverge. This means that for any growth order with divergent partial sums, we can, in fact, construct a *strictly decreasing infinite sequence* of growth orders starting with the given growth order, each of whose partial sums diverges. The following growth orders are familiar ones from calculus:

$$\cdots < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2\mathfrak{l}_3)^{-1} < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2)^{-1} < (\mathfrak{n}\mathfrak{l})^{-1} < \mathfrak{n}^{-1}$$

where the partial sums of the sequence $(\mathfrak{n}\mathfrak{l} \cdots \mathfrak{l}_m)^{-1}$ diverge with a growth order of $\mathfrak{l}_{m+1}$. This provokes another question. Sure, we know that there is no *least* diverging growth order, but maybe some sequence of growth orders like the above "covers" all growth orders whose partial sums diverge. For instance, perhaps we can say that every growth order $\mathfrak{a}$ whose partial sums diverge falls above some growth order from the above list, so that no growth order diverges more slowly than *all* of the growth orders $(\mathfrak{n}\mathfrak{l} \cdots \mathfrak{l}_m)^{-1}$. Is this true?

Proposition 39. *Given any infinite descending sequence of growth orders*

$$\cdots < \mathfrak{a}_3 < \mathfrak{a}_2 < \mathfrak{a}_1$$

such that $\Sigma\mathfrak{a}_i > \mathbf{1}$ *for all* $i \in \mathbb{N}$, *there always exists some growth order* $\mathfrak{c}$ *such that* $\mathfrak{c} < \mathfrak{a}_i$ *for all* $i \in \mathbb{N}$ *and yet* $\Sigma\mathfrak{c} > \mathbf{1}$.

Proof. Let's start by choosing infinitely many sequences $(a_n^{(i)}) \in \mathfrak{a}_i$ from the given sequence of growth orders. Because $\mathfrak{a}_{i+1} < \mathfrak{a}_i$ for each $i \in \mathbb{N}$, there exists some sequence of constants C_i such that

$$\frac{a_n^{(i+1)}}{a_n^{(i)}} \leq C_i$$

for all $n \in \mathbb{N}$, for each $i \in \mathbb{N}$. This inequality implies that

$$\frac{a_n^{(i+1)}}{a_n^{(j)}} \leq C_i C_{i-1} \cdots C_j$$

for all $n \in \mathbb{N}$, for all $j \leq i$. Denote the constant $C_i C_{i-1} \cdots C_j$ by $B_{i,j}$.

Let us now define another class of sequences $(b_n^{(i)})$ as follows: set $(b_n^{(1)}) = (a_n^{(1)})$, and

$$b_n^{(i+1)} = \frac{a_n^{(i+1)}}{\max(B_{i,1}, B_{i,2}, \cdots, B_{i,i})}$$

so that we have $b_n^{(i)} \leq b_n^{(j)}$ for all $n \in \mathbb{N}$ and $j \leq i$. In essence, we have normalized the sequences $(a_n^{(i)})$ so that each sequence $(b_n^{(i)})$ has the same growth order of $\mathfrak{a}_i$ for a fixed value of i, but $b_n^{(i)}$ is a decreasing function of i. In other words, if we arrange these sequences in a table:

$$\begin{array}{cccccc}
b_1^{(1)} & b_2^{(1)} & b_3^{(1)} & \cdots & b_n^{(1)} & \cdots \\
b_1^{(2)} & b_2^{(2)} & b_3^{(2)} & \cdots & b_n^{(2)} & \cdots \\
b_1^{(3)} & b_2^{(3)} & b_3^{(3)} & \cdots & b_n^{(3)} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_1^{(i)} & b_2^{(i)} & b_3^{(i)} & \cdots & b_n^{(i)} & \cdots
\end{array}$$

then the sequence along row i has growth order $\mathfrak{a}_i$, and the sequence down column n is monotone decreasing.

Since each sequence $b_n^{(i)}$ diverges, for any given $M \in \mathbb{R}^+$, there exists some index m such that the sum of the first m terms of the sequence exceeds M. Therefore, we may define a more general function $\text{ind}_M(i)$ as follows: let $\text{ind}_M(i)$ be the smallest value of m such that the sum of the first m terms of the sequence $(b_n^{(i)})$ exceeds M. Notice that $\text{ind}_M(i) \leq \text{ind}_M(j)$ when $i \leq j$, since $b_n^{(i)} \geq b_n^{(j)}$ for $i \leq j$. Informally, for sequences on lower rows of the table, it takes their partial sums longer to reach large values.

Finally, let's define a new sequence (c_n), which will in some sense "diagonalize" over the above family of sequences. Define c_n piecewise as follows:

$$c_n = \begin{cases} b_n^{(1)} & \text{if } n \leq \text{ind}_1(1) \\ b_n^{(i+1)} & \text{when } \text{ind}_i(i) < n \leq \text{ind}_{i+1}(i+1) \end{cases}$$

notice that these cases well-define c_n because $\mathrm{ind}_i(i)$ is a monotone increasing sequence of i, and therefore every $n \geq \mathrm{ind}_i(i)$ falls in the interval of integers $(\mathrm{ind}_i(i), \mathrm{ind}_{i+1}(i+1)]$ for exactly one value of $i \in \mathbb{N}$. (Some of these intervals are empty, namely the ones where $\mathrm{ind}_i(i) = \mathrm{ind}_{i+1}(i+1)$.) At this point we just need to show that (c_n) satisfies the desired properties of having growth order less than each $\mathfrak{a}_i$, and having divergent partial sums.

First of all, notice that for all $n > \mathrm{ind}_i(i)$, we have that $c_n \leq b_n^{(i)}$, since for these values of n we will have $c_n = b_n^{(j)}$ for some values $j > i$, and we have already shown that $b_n^{(j)} \leq b_n^{(i)}$ for $j \geq i$. Thus, since c_n is bounded above by $b_n^{(i)}$ for all but finitely many values of n. This means that, if $\mathfrak{c} = [(c_n)]$, we have $\mathfrak{c} \leq \mathfrak{a}_i$ for each i, and hence $\mathfrak{c} \leq \mathfrak{a}_{i+1} < \mathfrak{a}_i$ and $\mathfrak{c} < \mathfrak{a}_i$ for each $i \in \mathbb{N}$, as claimed.

Finally, consider the sum of the first $\mathrm{ind}_i(i)$ values of c_n. For each $n \leq \mathrm{ind}_i(i)$, we have that $c_n = b_n^{(j)}$ for some $j < i$, meaning that $c_n \geq b_n^{(i)}$ for each $n \leq \mathrm{ind}_i(i)$. But, by the definition of $\mathrm{ind}_i(i)$, we have that the sum $b_1^{(i)} + \cdots + b_{\mathrm{ind}_i(i)}^{(i)} \geq i$, meaning that we also have $c_1 + \cdots + c_{\mathrm{ind}_i(i)} \geq i$. Thus, the partial sums of (c_n) are unbounded, and $\Sigma\mathfrak{c} > \mathbf{1}$, as desired. □

Again, our question is answered negatively. Given any descending sequence of growth orders with divergent partial sums, there exists a growth orders with partial sums that diverge even more slowly.

What does this look like in terms of the familiar sequence of growth orders

$$\cdots < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2\mathfrak{l}_3)^{-1} < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2)^{-1} < (\mathfrak{n}\mathfrak{l})^{-1} < \mathfrak{n}^{-1}$$

mentioned earlier? Well, let's repeat the construction for this family of growth orders. Our family of sequences $(a_n^{(i)})$ can be given as follows:

$$\begin{aligned} a_n^{(1)} &= n^{-1} \\ a_n^{(2)} &= n^{-1}(1 + \log_2 n)^{-1} \\ a_n^{(3)} &= n^{-1}(1 + \log_2 n)^{-1}\big(1 + \log_2(1 + \log_2 n)\big)^{-1} \\ &\cdots \end{aligned}$$

For these sequences, we already have $a_n^{(j)} \leq a_n^{(i)}$ for all $i \leq j$, so we don't even need to bother with normalizing our sequences. The partial sums of $a_n^{(i)}$ are on the order $\mathfrak{l}_i$, meaning that $\mathrm{ind}_i(i)$ will look something like ${}^{i}2$, where the left superscript denotes tetration - that is, a power tower consisting of i many 2s. Therefore, if we let $\mathrm{slog}_2(n)$ denote the smallest natural number i such that $\mathrm{ind}_i(i) < i$, then our diagonalizing sequence (c_n) could be given by

$$c_n = n^{-1} \cdot (1 + \log_2 n)^{-1} \cdot \big(1 + \log_2(1 + \log_2 n)\big)^{-1} \cdot \cdots \cdot \Big(\overbrace{1 + \log_2\big(\cdots(1 + \log_2 n)\cdots\big)}^{\mathrm{slog}_2(n) \text{ nested logarithms}}\Big)^{-1}$$

This sequence decays faster than each growth order $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$, yet its partial sums still diverge. (Note: the name slog_2 is chosen as a reference to the so-called "super-logarithm", which is sometimes defined as an analogue of the logarithm for tetration rather than exponentiation.)

Now consider the proof of the previous theorem, this time paying attention to the special case in which the ratios $\mathfrak{a}_i/\mathfrak{a}_j$ are monotone. Because it is often convenient for us to deal with growth orders whose pairwise quotients are monotone, we would like to be able to construct a growth order $\mathfrak{c}$ with divergent sums not only satisfying $\mathfrak{c} < \mathfrak{a}_i$ for each i, but also making the quotients $\mathfrak{a}_i/\mathfrak{c}$ monotone. It is not hard to see, however, that the construction from the above proof will result in $\mathfrak{c}$ having monotone quotients with each of the $\mathfrak{a}_i$. This can be shown by choosing the sequences $(a_n^{(i)})$ (and therefore also $(b_n^{(i)})$) to have monotone quotients, and then noticing that each of the sequences $(c_n/b_n^{(i)})$ is *eventually* monotone, specifically monotone for all $n > \mathrm{ind}_{i-1}(i-1)$. That is,

$$\frac{c_{n+1}}{b_{n+1}^{(i)}} \leq \frac{c_n}{b_n^{(i)}}$$

whenever $n > \mathrm{ind}_{i-1}(i-1)$. When n does not take the form $n = \mathrm{ind}_m(m)$, this inequality follows from the fact that the sequences $(b_n^{(j)}/b_n^{(i)})$ are monotone decreasing for $j \geq i$, since c_n is piecewise defined by these sequences on successive intervals. When $n = \mathrm{ind}_m(m)$, the inequality follows from this combined with the fact that $b_n^{(j+1)} \leq b_n^{(j)}$ for all j. Hence, we have the following refinement of the previous proposition:

Proposition 40. *Given an infinite descending sequence of growth orders*

$$\cdots < \mathfrak{a}_3 < \mathfrak{a}_2 < \mathfrak{a}_1 < \mathbf{1}$$

such that $\mathfrak{a}_i/\mathfrak{a}_j$ is monotone and $\Sigma\mathfrak{a}_i > \mathbf{1}$ for any $i, j \in \mathbb{N}$, there always exists some growth order $\mathfrak{c}$ such that $\mathfrak{a} < \mathfrak{a}_i$ with $\mathfrak{a}_i/\mathfrak{c}$ monotone for all $i \in \mathbb{N}$, and yet $\Sigma\mathfrak{c} > \mathbf{1}$.

5. Composition and inverses

5.1. Definition of the composite

Given a sequence (a_n), we might want to examine the ways in which its growth order can change when it is reindexed. For instance, we may want to consider subsequences like (a_{2n}) - which, as we proved earlier, has the same growth order as (a_n) given moderate growth - or (a_{n^2}). These subsequences *accelerate* the growth or decay of the sequence (a_n), but we might also consider subsequences like $(a_{\lfloor\sqrt{n}\rfloor})$ that "slow down" the original sequence.

In general, we may want to consider (a_{b_n}) for an arbitrary indexing sequence (b_n). However, this only makes sense when b_n is a sequence of natural numbers, since a_n is only defined for $n \in \mathbb{N}$. Hence, if we want to define the composite of two sequences $\alpha \circ \beta = (a_{b_n})$, we need (b_n) to consist of natural numbers.

Definition 14. *Given sequences $\alpha \in \mathcal{S}(\mathbb{R}^+)$ and $\beta \in \mathcal{S}(\mathbb{N})$, define their* ***composite****, denoted by $\alpha \circ \beta$, to be the sequence (a_{b_n}), and β is called the* ***indexing sequence****.*

However, even if (b_n) is not a sequence of natural numbers, there may still exist a sequence of natural numbers *of the same growth order* which could be used as a sequence of indices in place of (b_n). In fact, the following proposition proves that such a sequence exists whenever $\mathfrak{b} \geq \mathbf{1}$.

Proposition 41. *For every sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha] \geq \mathbf{1}$, there exists a sequence of natural numbers $\beta \in \mathcal{S}(\mathbb{N})$ such that $[\alpha] = [\beta]$.*

Proof. Let $[\alpha] \geq \mathbf{1}$. Then we shall show that the sequence $\beta = (b_n) \in \mathcal{S}(\mathbb{N})$ defined by $b_n = \lceil a_n \rceil$ has the same growth order as α. Since $\lceil x \rceil - x \in [0, 1)$ for all $x \in \mathbb{R}^+$, it follows that $b_n - a_n \in [0, 1)$ and therefore

$$a_n \leq b_n \leq a_n + 1$$

for all $n \in \mathbb{N}$. Since $[\alpha] \geq \mathbf{1}$, we have that $a_n \geq C$ for some $C \in \mathbb{R}^+$, meaning that $a_n + 1 \leq (1 + C^{-1})a_n$, and therefore

$$a_n \leq b_n \leq (1 + C^{-1})a_n$$

for all $n \in \mathbb{N}$, proving that $[\alpha] = [\beta]$ as claimed. □

Now that we can define the composite of two sequences α, β with $[\beta] \geq \mathbf{1}$, we'd like to define it on growth orders as well. The most natural definition would be to let $\mathfrak{a} \circ \mathfrak{b} = [\alpha] \circ [\beta] = [\alpha \circ \beta]$.

However, we must first show that this operation is well-defined for the class of growth orders that we are concerned with - namely, the moderate ones.

Proposition 42. *Let $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ and $\beta, \beta' \in \mathcal{S}(\mathbb{N})$. If $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, and α, α' exhibit moderate growth, then $[\alpha \circ \beta] = [\alpha' \circ \beta']$.*

NEW (BETTER) PROOF:

Proof. It suffices to show the following two facts:

1. If $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ are moderate with $\alpha \sim \alpha'$ and $\beta \in \mathcal{S}(\mathbb{N})$, then $\alpha \circ \beta \sim \alpha' \circ \beta$.
2. If $\alpha \in \mathcal{S}(\mathbb{R}^+)$ is moderate and $\beta, \beta' \in \mathcal{S}(\mathbb{N})$ with $\beta \sim \beta'$, then $\alpha \circ \beta \sim \alpha \circ \beta'$.

In other words, we are showing that the growth order of $\alpha \circ \beta$ depends only on the respective growth orders of α and β.

The former claim (1) is trivial to show: because $\alpha \sim \alpha'$, we have that there exist constants $C_1, C_2 > 0$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

for all $n \in \mathbb{N}$. Because this holds for all natural numbers n, we may replace n with b_n, obtaining the inequality

$$C_1 a_{b_n} \leq a'_{b_n} \leq C_2 a_{b_n}$$

which simply states that $\alpha \circ \beta \sim \alpha' \circ \beta$ by definition.

The latter claim is a little more cumbersome to prove. Let α, β, β' be given as in (2), and let $C_1, C_2 > 0$ be constants such that

$$C_1 b_n \leq b'_n \leq C_2 b_n$$

for all $n \in \mathbb{N}$. This implies the following weaker inequality, since $\lceil x \rceil \geq x$ and $\lceil x^{-1} \rceil^{-1} \leq x$ for all $x > 0$:

$$\lceil C_1^{-1} \rceil^{-1} b_n \leq b'_n \leq \lceil C_2 \rceil b_n$$

Therefore, we have natural numbers $K_1 = \lceil C_1^{-1} \rceil$ and $K_2 = \lceil C_2 \rceil$ such that

$$K_1^{-1} b_n \leq b'_n \leq K_2 b_n$$

Now, because b'_n is an integer, and $\lceil K_1^{-1} b_n \rceil$ is the smallest integer greater than or equal to $K_1^{-1} b_n$, we have that

$$\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 b_n$$

Further, notice that $b_n \geq K_1 \lceil K_1^{-1} b_n \rceil$, since $\lceil K^{-1} b_n \rceil = K^{-1} b_n + \delta$ for some $\delta \in [0, 1)$, meaning that $K_1 \lceil K_1^{-1} b_n \rceil = b_n + K_1 \delta$. Thus, we may loosen the upper bound by replacing b_n with $K_1 \lceil K_1^{-1} b_n \rceil$:

$$\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 K_1 \lceil K_1^{-1} b_n \rceil$$

Now we shall make use of the moderateness property of α, which we will use to procure two constants D_1, D_2. First, we may let $D_1 > 0$ be a constant such that

$$a_m \leq D_1 a_n$$

for all $n \leq m \leq K_2 K_1 n$. Secondly, we may let $D_2 > 0$ be a constant such that $a_m \leq D_2 a_n$ for all $n \leq m \leq (K_1 + 1)n$. We shall use these definitions to procure a chain of inequalities culminating in the inequality $a_{b'_n} \leq D_1 D_2^2 a_{b_n}$. First of all, we have that

$$a_{b'_n} \leq D_1 a_{\lceil K_1^{-1} b_n \rceil}$$

by the definition of D_1, and because $\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 K_1 \lceil K_1^{-1} b_n \rceil$ as proven earlier. (We are using the definition of D_1, but replacing m with b'_n and replacing n with $\lceil K_1^{-1} b_n \rceil$.) Next, we have the inequality

$$D_1 a_{\lceil K_1^{-1} b_n \rceil} \leq D_1 D_2 a_{K_1 \lceil K_1^{-1} b_n \rceil}$$

by the definition of D_2, and because $\lceil K_1^{-1} b_n \rceil \leq K_1 \lceil K_1^{-1} b_n \rceil \leq (K_1 + 1)\lceil K_1^{-1} b_n \rceil$. (This time, we are using the definition of D_2 while replacing m with $K_1 \lceil K_1^{-1} b_n \rceil$ and replacing n with $\lceil K_1^{-1} b_n \rceil$.) Finally, we have the inequality

$$D_1 D_2 a_{K_1 \lceil K_1^{-1} b_n \rceil} \leq D_1 D_2^2 a_{b_n}$$

which follows from the definition of D_2 and the incredibly weak inequality $b_n \leq K_1 \lceil K_1^{-1} b_n \rceil \leq (K_1 + 1) b_n$. (We are using the definition of D_2 again and replacing m with $K_1 \lceil K_1^{-1} b_n \rceil$ and replacing n with b_n.) Thus, we have established the following chain of 3 inequalities:

$$a_{b'_n} \leq D_1 a_{\lceil K_1^{-1} b_n \rceil} \leq D_1 D_2 a_{K_1 \lceil K_1^{-1} b_n \rceil} \leq D_1 D_2^2 a_{b_n}$$

which, at least, tells us that $a_{b'_n} \leq D_1 D_2^2 a_{b_n}$, or that $[\alpha \circ \beta'] \leq [\alpha \circ \beta]$. However, since β, β' were completely arbitrary, we have by symmetry that $[\alpha \circ \beta] \leq [\alpha \circ \beta']$ as well, meaning that $\alpha \circ \beta \sim \alpha \circ \beta'$. This completes the proof of (2). □

OLD (SLOPPY) PROOF:

Proof. If we are given that $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then we may let $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ be constants such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

$$C_3 b_n \leq b'_n \leq C_4 b_n$$

for all $n \in \mathbb{N}$. The second inequality above implies the weaker inequality

$$\lceil C_3^{-1} \rceil^{-1} b_n \leq b'_n \leq \lceil C_4 \rceil b_n$$

We therefore have natural numbers $K_1 = \lceil C_3^{-1} \rceil, K_2 = \lceil C_4 \rceil$ such that

$$K_1^{-1} b_n \leq b'_n \leq K_2 b_n$$

for all n. Furthermore, since b_n is always an integer,

$$\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 b_n$$

and consequently

$$\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 K_1 \lceil K_1^{-1} b_n \rceil$$

for all n. By the moderate growth property of a and the above inequality, there exist constants $D_1, D_2 \in \mathbb{R}^+$ such that $a_{b'_n} \leq D_1 a_{\lceil K_1^{-1} b_n \rceil}$ for all $n \in \mathbb{N}$, and $a_n \leq D_2 a_{(K_1+1)n}$ for all $n \in \mathbb{N}$. Then we have

$$a'_{b'_n} \leq D_1 a'_{\lceil K_1^{-1} b_n \rceil} \leq D_1 D_2 a'_{K_1 \lceil K_1^{-1} b_n \rceil} \leq D_1 D_2^2 a'_{b_n}$$

since $K_1 \lceil K_1^{-1} b_n \rceil$ is always between b_n and $(K_1 + 1) b_n$. From this, we have that

$$a'_{b'_n} \leq C_2 D_1 D_2^2 a_{b_n}$$

and therefore $a'_{b'_n} = O(a_{b_n})$. Since the same line of reasoning also shows that $a_{b_n} = O(a'_{b'_n})$, we have that $(a_{b_n}) \sim (a'_{b'_n})$ and therefore $[\alpha \circ \beta] = [\alpha' \circ \beta']$ as claimed. □

This proves that the composition operation can be well-defined on *growth orders* of sequences, not just individual sequences. In particular, we may define the composition of growth orders $\mathfrak{a} \circ \mathfrak{b}$ whenever $\mathfrak{b} \geq \mathbf{1}$, by choosing an arbitrary sequence $\alpha \in \mathfrak{a}$ and an indexing sequence of natural numbers $\beta \in \mathfrak{b}$ (whose existence is guaranteed by the fact that $\mathfrak{b} \geq \mathbf{1}$) and considering the growth order $[\alpha \circ \beta]$. This is well-defined because this equivalence class is independent of the choice of α and β, as proven by the previous proposition.

Definition 15. *Given growth orders* $\mathfrak{a}, \mathfrak{b}$ *with* $\mathfrak{a}$ *moderate and* $\mathfrak{b} \geq 1$*, define their* ***composite*** $\mathfrak{a} \circ \mathfrak{b}$ *as the equivalence class* $[\alpha \circ \beta]$*, where* $\alpha \in \mathfrak{a}$ *and* $\beta \in \mathfrak{b} \cap \mathcal{S}(\mathbb{N})$ *are arbitrary.*

Note that the restriction to moderate growth sequences is necessary. Consider the sequence $a_n = 2^{n^2}$ and the two indexing sequences $b_n = n$ and $b'_n = n + 1$. Clearly we have $(b_n) \sim (b'_n)$ and therefore $\mathfrak{b} = [(b_n)] = [(b'_n)]$. However, $a_{b_n} = 2^{n^2}$ has a strictly lesser growth order than $a_{b'_n} = 2^{(n+1)^2}$. Hence $\mathfrak{a} \circ \mathfrak{b}$ is not well-defined, as we have two $\beta, \beta' \in \mathfrak{b}$ for which $[\alpha \circ \beta] \neq [\alpha \circ \beta']$.

If we consider growth orders with properties that make them amenable to both left- and right-composition, we can form subsets of $\mathcal{S}(\mathbb{R}^+)$ that are closed under composition. Such subsets sometimes carry the structure of a *monoid*, since the binary operation of composition is associative.

Proposition 43. *If* $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)$ *consists of moderate growth orders* ≥ 1 *and is closed under composition, then it is a semigroup under composition. If it contains* $\mathfrak{n}$*, then it is a monoid under composition.*

Since the properties of moderateness and monotonicity have proven useful to us so far, it is worth showing that composition of growth orders preserves these properties.

Proposition 44. *If $\mathfrak{a}$ and $\mathfrak{b} \geq 1$ are moderate, then $\mathfrak{a} \circ \mathfrak{b}$ is moderate.*

Proof. Let $(a_n) \in \mathfrak{a}$ be arbitrary, and let $(b_n) \in \mathfrak{b}$ be a sequence of positive integers. Fix some $k \in \mathbb{N}$. Because (b_n) is moderate, there exist constants C_1, C_2 such that

$$C_1 b_n \leq b_m \leq C_2 b_n$$

for all $m, n \in \mathbb{N}$ with $n \leq m \leq kn$. This implies that

$$\lceil C_1^{-1} \rceil^{-1} b_n \leq b_m \leq \lceil C_2 \rceil b_n$$

Now let (b'_n) be a sequence defined by $b'_n = \lceil C_1^{-1} \rceil b_n$, so that (b'_n) is a sequence of positive integers with the same growth order as (b_n), such that $\lceil C_1^{-1} \rceil^{-1} b'_n$ is a positive integer for all $n \in \mathbb{N}$. Then we also have

$$\lceil C_1^{-1} \rceil^{-1} b'_n \leq b'_m \leq \lceil C_2 \rceil b'_n$$

Now, because (a_n) is moderate, we have that:

1. There exist constants C_3, C_4 such that $C_3 a_n \leq a_m \leq C_4 a_n$ for all $n \leq m \leq \lceil C_1^{-1} \rceil n$
2. There exist constants C_5, C_6 such that $C_5 a_n \leq a_m \leq C_6 a_n$ for all $n \leq m \leq \lceil C_2 \rceil n$
3. There exist constants C_7, C_8 such that $C_7 a_n \leq a_m \leq C_8 a_n$ for all $n \leq m \leq \lceil C_1^{-1} \rceil \lceil C_2 \rceil n$

From the previous inequality derived for (b'_n), we know that, for all $m, n \in \mathbb{N}$ such that $n \leq m \leq kn$,

$$C_7 a_{\lceil C_1^{-1} \rceil^{-1} b'_n} \leq a_{b'_m} \leq C_8 a_{\lceil C_2 \rceil b'_n}$$

Further, from the definitions of the above constants, we have that

$$a_{\lceil C_1^{-1} \rceil^{-1} b'_n} \geq C_4^{-1} a_{b'_n}$$

and

$$a_{\lceil C_2 \rceil b'_n} \leq C_6 a_{b'_n}$$

so that

$$C_4^{-1} C_7 a_{b'_n} \leq a_{b'_m} \leq C_6 C_8 a_{b'_n}$$

for all $n \leq m \leq kn$. It follows that the sequence $(a_{b'_n})$ is moderate, and since $(b_n) \sim (b'_n)$ have the same growth order $\mathfrak{b}$, we have that $\mathfrak{a} \circ \mathfrak{b}$ is moderate, as claimed. □

Proving that composition preserves monotonicity is much more straightforward:

Proposition 45. *If $\mathfrak{a}$ is monotone and $\mathfrak{b} \geq 1$ are monotone, then $\mathfrak{a} \circ \mathfrak{b}$ is monotone.*

Proof. This proof is almost trivial. Let $(a_n) \in \mathfrak{a}$ be monotone and $(b_n) \in \mathfrak{b}$ be a monotone increasing sequence of positive integers. If (a_n) is monotone increasing, then $i \le j$ implies $b_i \le b_j$ and $a_{b_i} \le a_{b_j}$, so that (a_{b_n}) is also monotone increasing. If (a_n) is monotone decreasing, then $i \le j$ implies $b_i \le b_j$ and $a_{b_i} \ge a_{b_j}$, so that (a_{b_n}) is also monotone decreasing. In either case, (a_{b_n}) is monotone, so we have that $\mathfrak{a} \circ \mathfrak{b}$ is monotone. □

5.2. Arithmetic and inequalities

Here are some elementary properties of composition, and its interactions with other operations on growth orders:

Proposition 46. *The following equalities hold for growth orders* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *whenever the stated composites are defined:*

- $(\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{b} = \mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c})$
- $(\mathfrak{a} + \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ \mathfrak{b} + \mathfrak{a} \circ \mathfrak{c}$
- $\mathfrak{a}\mathfrak{b} \circ \mathfrak{c} = (\mathfrak{a} \circ \mathfrak{c})(\mathfrak{b} \circ \mathfrak{c})$
- $\mathfrak{a}^p = \mathfrak{n}^p \circ \mathfrak{a}$

Proof. Trivial. □

Proposition 47. *The following equalities hold for growth orders* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *whenever the stated composites are defined:*

- $\mathfrak{a} \le \mathfrak{b} \implies \mathfrak{a} \circ \mathfrak{c} \le \mathfrak{b} \circ \mathfrak{c}$
- $\mathfrak{a} \ge 1$ *monotone,* $\mathfrak{b} \le \mathfrak{c} \implies \mathfrak{a} \circ \mathfrak{b} \le \mathfrak{a} \circ \mathfrak{c}$
- $\mathfrak{a} \le 1$ *monotone,* $\mathfrak{b} \le \mathfrak{c} \implies \mathfrak{a} \circ \mathfrak{b} \ge \mathfrak{a} \circ \mathfrak{c}$

Proof. For the first claim, suppose $\mathfrak{a} \le \mathfrak{b}$ and let $(c_n) \in \mathfrak{c}$ be a sequence of positive integers. If $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$ such that a_n/b_n is bounded above, then we clearly have that a_{c_n}/b_{c_n} is bounded above (by the same upper bound). The sequences a_{c_n} and b_{c_n} have growth orders $\mathfrak{a} \circ \mathfrak{c}$ and $\mathfrak{b} \circ \mathfrak{c}$ respectively, so we have that $\mathfrak{a} \circ \mathfrak{c} \le \mathfrak{b} \circ \mathfrak{c}$.

For the second claim, suppose that $(a_n) \in \mathfrak{a}$ is monotone increasing and $(b_n) \in \mathfrak{b}$, $(c_n) \in \mathfrak{c}$. If $\mathfrak{b} \le \mathfrak{c}$, then b_n/c_n is bounded above by some positive constant C. If we define another sequence $b'_n = b_n/C$, then we have that $b'_n \le c_n$ for all $n \in \mathbb{N}$ and $(b'_n) \in \mathfrak{b}$. Since (a_n) is monotone increasing we have that $a_{b'_n} \le a_{c_n}$ and therefore $\mathfrak{a} \circ \mathfrak{b} \le \mathfrak{a} \circ \mathfrak{c}$. The argument is much the same for the third claim, except that (a_n) will be monotone decreasing. □

Note that the restriction to monotone growth orders $\mathfrak{a}$ in the latter two claims is essential, for neither implication is necessarily true for non-monotone growth orders $\mathfrak{a}$. Consider, for instance, the sequence (a_n) defined by

$$a_n = n^{1+\sin \log \log n}$$

A similar sequence was considered in a previous counterexample in section 4.2, page 24 , and the same technique can be applied here to show that a_n is monotone. Consider also the indexing sequences (b_n), (c_n) given by $b_n = n$ and $c_n = \lfloor n^p \rfloor$, where $p = e^{e^\pi}$. We have that $|c_n - n^p| \leq 1$, and therefore

$$|\log\log c_n - \log\log n| = \pi + o(1)$$

which means that, since the sine function is continuous,

$$\sin\log\log c_n = -\sin\log\log b_n + o(1)$$

which implies that

$$\frac{a_{b_n}}{a_{c_n}} = n^{2\sin\log\log n} \cdot n^{o(1)}$$

This sequence is not comparable to $\mathbf{1}$, since it has subsequences tending to 0 and to ∞. Hence, we have that $\mathfrak{a} \circ \mathfrak{b} \perp \mathfrak{a} \circ \mathfrak{c}$, showing that $\mathfrak{a} \geq 1$ and $\mathfrak{b} \leq \mathfrak{c}$ do not necessarily imply that $\mathfrak{a} \circ \mathfrak{b} \leq \mathfrak{a} \circ \mathfrak{c}$.

5.3. Absorption

Definition 16. *Given growth orders $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{a} \circ \mathfrak{b}$ is defined, we say that $\mathfrak{a}$ **absorbs** $\mathfrak{b}$ if $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{a}$.*

Proposition 48. *If $\mathfrak{a}$ absorbs $\mathfrak{b}$ and $\mathfrak{b}$ absorbs $\mathfrak{c}$, then $\mathfrak{a}$ absorbs $\mathfrak{c}$.*

Proof. Given that the compositions $\mathfrak{a} \circ \mathfrak{b}$ and $\mathfrak{b} \circ \mathfrak{c}$ are defined, with $\mathfrak{a}$ absorbing $\mathfrak{b}$ and $\mathfrak{b}$ absorbing $\mathfrak{c}$, we have that

$$\mathfrak{a} \circ \mathfrak{c} = (\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c}) = \mathfrak{a} \circ \mathfrak{b} = \mathfrak{a}$$

and therefore $\mathfrak{a} \circ \mathfrak{c} = \mathfrak{a}$ and $\mathfrak{a}$ absorbs $\mathfrak{c}$. □

Proposition 49. *If $\mathfrak{a}, \mathfrak{b}$ are moderate growth orders such that $\mathfrak{b}/\mathfrak{a}$ is monotone, and $\mathfrak{c} \geq \mathfrak{n}$, then it follows that*

$$\mathfrak{a} \leq \mathfrak{b} \implies \frac{(\Sigma\mathfrak{a}) \circ \mathfrak{c}}{\Sigma\mathfrak{a}} \leq \frac{(\Sigma\mathfrak{b}) \circ \mathfrak{c}}{\Sigma\mathfrak{b}}$$

Proof. Let $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b}/\mathfrak{a}$ be monotone. Then we may choose a monotone increasing sequence $(r_n) \in \mathfrak{b}/\mathfrak{a}$, and sequences $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ such that $b_n = r_n a_n$ for all $n \in \mathbb{N}$. Further, we may choose some $(c_n) \in \mathfrak{c}$ such that $c_n \geq n$ for all $n \in \mathbb{N}$. Then notice that, since (r_n) is monotone increasing, we have the following two inequalities:

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k a_k \leq r_n \sum_{k=1}^{n} a_k$$

$$\sum_{k=n+1}^{c_k} b_k = \sum_{k=n+1}^{c_n} r_k a_k \geq r_n \sum_{k=n+1}^{c_n} a_k$$

Therefore, we have the following chain of inequalities:

$$\left(\sum_{k=n+1}^{c_n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq \left(\sum_{k=n+1}^{c_n} a_k\right) \cdot r_n \left(\sum_{k=1}^{n} a_k\right) \leq \left(\sum_{k=n+1}^{c_n} b_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

and therefore

$$\left(\sum_{k=n+1}^{c_n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq \left(\sum_{k=n+1}^{c_n} b_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

or, equivalently,

$$\frac{\sum_{k=n+1}^{c_n} a_k}{\sum_{k=1}^{n} a_k} \leq \frac{\sum_{k=n+1}^{c_n} b_k}{\sum_{k=1}^{n} b_k}$$

or, by adding 1 to both sides of this inequality, we have

$$\frac{\sum_{k=1}^{c_n} a_k}{\sum_{k=1}^{n} a_k} \leq \frac{\sum_{k=1}^{c_n} b_k}{\sum_{k=1}^{n} b_k}$$

which implies that

$$\frac{(\Sigma\mathfrak{a}) \circ \mathfrak{c}}{\Sigma\mathfrak{a}} \leq \frac{(\Sigma\mathfrak{b}) \circ \mathfrak{c}}{\Sigma\mathfrak{b}}$$

as desired. □

The lemma above allows us to prove a sort of "squeezing" property of absorption.

Proposition 50. *Let $\mathfrak{a}, \mathfrak{b}$ be moderate and monotone growth orders such that $\mathfrak{b}/\mathfrak{a}$ is monotone and $\mathbf{1} \leq \mathfrak{a} \leq \mathfrak{b}$. If $\mathfrak{c} \geq \mathfrak{n}$ and $\Sigma\mathfrak{b}$ absorbs $\mathfrak{c}$, then $\Sigma\mathfrak{a}$ also absorbs $\mathfrak{c}$.*

Proof. If $\Sigma\mathfrak{b}$ absorbs $\mathfrak{c}$, then $(\Sigma\mathfrak{b}) \circ \mathfrak{c} = \Sigma\mathfrak{b}$, and therefore

$$\frac{(\Sigma\mathfrak{b}) \circ \mathfrak{c}}{\Sigma\mathfrak{b}} = \mathbf{1}$$

Similarly, since $\mathbf{1}$ absorbs every growth order, we have

$$\frac{\mathbf{1} \circ \mathfrak{c}}{\mathbf{1}} = \mathbf{1}$$

Now, since $\mathbf{1} \leq \mathfrak{a} \leq \mathfrak{b}$ and the ratios $\mathfrak{a}/\mathfrak{n}^{-2}$ and $\mathfrak{b}/\mathfrak{a}$ are monotone (where $\mathbf{1} = \Sigma\mathfrak{n}^{-2}$), we have from the above proposition that

$$\frac{\mathbf{1} \circ \mathfrak{c}}{\mathbf{1}} \leq \frac{(\Sigma\mathfrak{a}) \circ \mathfrak{c}}{\Sigma\mathfrak{a}} \leq \frac{(\Sigma\mathfrak{b}) \circ \mathfrak{c}}{\Sigma\mathfrak{b}}$$

or
$$\mathbf{1} \leq \frac{(\Sigma \mathfrak{a}) \circ \mathfrak{c}}{\Sigma \mathfrak{a}} \leq \mathbf{1}$$
which implies that
$$\frac{(\Sigma \mathfrak{a}) \circ \mathfrak{c}}{\Sigma \mathfrak{a}} = \mathbf{1}$$
and therefore $(\Sigma \mathfrak{a}) \circ \mathfrak{c} = \Sigma \mathfrak{a}$, meaning that $\Sigma \mathfrak{a}$ also absorbs $\mathfrak{c}$ as claimed. □

Exercise 5 If $f : \mathbb{N} \to \mathbb{N}$ is an increasing function with the property that $f(a_n) = O(f(n))$ for all $O(n \log n)$ sequences $(a_n) \subset \mathbb{N}$, show that this bound also holds for all $O(n \log^{42} n)$ sequences $(a_n) \subset \mathbb{N}$.

The following proposition shows that it's easy to take partial sums of growth orders that are very "absorbant":

Proposition 51. *If $\mathfrak{a}$ is monotone and absorbs some $\mathfrak{b} < \min(\mathfrak{n}, \mathfrak{a}\mathfrak{n})$, then $\Sigma \mathfrak{a} = \mathfrak{n}\mathfrak{a}$.*

Proof. Let $(a_n) \in \mathfrak{a}$ be monotone, and choose $(b_n) \in \mathfrak{b}$ to be a sequence of integers such that $b_n \leq n$ for all $n \in \mathbb{N}$.

First suppose (a_n) is monotone increasing. Then we have that
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{b_n} a_i + \sum_{i=b_n+1}^{n} a_i \geq (n - b_n + 1) a_{b_n} \sim n a_n$$
since $\mathfrak{b} < \mathfrak{n}$ and $a_{b_n} \sim a_n$. We also clearly have that
$$\sum_{i=1}^{n} a_i \leq n a_n$$
because (a_n) is monotone increasing. Thus, we have $\mathfrak{n}\mathfrak{a} \leq \Sigma \mathfrak{a} \leq \mathfrak{n}\mathfrak{a}$, and therefore $\Sigma \mathfrak{a} = \mathfrak{n}\mathfrak{a}$.

Now suppose that (a_n) is monotone decreasing. This time we have that
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{b_n} a_i + \sum_{i=b_n+1}^{n} a_i \leq a_1 b_n + (n - b_n) a_{b_n} \sim n a_n$$
because $(n - b_n) a_{b_n} \sim n a_n$ (since $\mathfrak{b} < \mathfrak{n}$), and $a_1 b_n + n a_n \sim n a_n$ (since $\mathfrak{b} < \mathfrak{a}\mathfrak{n}$). On the other hand, we clearly have
$$\sum_{i=1}^{n} a_i \geq n a_n$$
because (a_n) is monotone decreasing. Thus, we again have $\mathfrak{n}\mathfrak{a} \leq \Sigma \mathfrak{a} \leq \mathfrak{n}\mathfrak{a}$ and therefore $\Sigma \mathfrak{a} = \mathfrak{n}\mathfrak{a}$. □

5.4. Inverses and cancellation

Definition 17. *Given growth orders $\mathfrak{a}$ and $\mathfrak{b}$, if $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{n}$, then we say that $\mathfrak{b}$ is a **right inverse** of $\mathfrak{a}$, and that $\mathfrak{a}$ is a **left inverse** of $\mathfrak{b}$.*

A right inverse of a growth order can be thought of as a way of reindexing sequences of that growth order in such a way that they exhibit linear growth. A left inverse, on the other hand, can be thought of as a sequence whose subsequences along a given reindexing exhibit linear growth.

We remarked earlier that subsets $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)$ consisting of moderate growth orders $\geq \mathbf{1}$ carry a *monoid structure* if they are closed under composition. From the monoid structure alone, we can deduce a couple basic properties of inverses of growth orders. For instance:

Proposition 52. *If a moderate growth order $\mathfrak{a} \geq \mathbf{1}$ has both a left-inverse $\mathfrak{b}$ and a right-inverse $\mathfrak{c}$, then $\mathfrak{b} = \mathfrak{c}$.*

Proof. Suppose that $\mathfrak{b} \circ \mathfrak{a} = \mathfrak{a} \circ \mathfrak{c} = \mathfrak{n}$. Then we have that $\mathfrak{b} \circ (\mathfrak{a} \circ \mathfrak{c}) = \mathfrak{b} \circ \mathfrak{n}$. The LHS of this equality is equal to $\mathfrak{c}$, since $\mathfrak{b} \circ \mathfrak{a} = \mathfrak{n}$ and $\mathfrak{n} \circ \mathfrak{c} = \mathfrak{c}$, making use of associativity. The RHS is equal to $\mathfrak{b} \circ \mathfrak{n} = \mathfrak{b}$. Thus, we have $\mathfrak{c} = \mathfrak{b}$ as claimed. □

Proposition 53. *If $\mathfrak{a}$ is moderate, then $\Sigma\mathfrak{a} > 1$ implies that $\Sigma\mathfrak{a}$ has a right-inverse.*

Proof. Let $\mathfrak{a}$ be a given moderate growth order, and let $\mathfrak{b} = \Sigma\mathfrak{a}$. This makes $\mathfrak{b}$ a monotone growth order, so that we may choose some monotone $(b_n) \in \mathfrak{b}$. Define the sequence (c_n) by letting c_n be the least natural number m such that $b_m \geq n$. (Such m always exists because $\mathfrak{b} > 1$ and b_n is therefore unbounded.) Notice that (c_n) is monotone and unbounded by this definition, because (b_n) cannot attain unboundedly large values in only finitely many terms.

Now consider the sequence (b_{c_n}). We will show that this sequence has growth order $\mathfrak{n}$, and that $\mathfrak{c} = [(c_n)]$ is therefore a right-inverse of $\mathfrak{b}$. Let $m \in \mathbb{N}$, and define $k = \lfloor b_m \rfloor$ and $l = \lfloor b_{m+1} \rfloor$. Then we have that

$$k \leq b_{c_k} < k + 1$$

because $b_{c_k} \geq k$ follows from the definition of c_k, and the upper bound follows from the fact that $c_k \leq m$ and therefore $b_{c_k} \leq b_m < k + 1$ by the monotonicity of (b_n). Now, using the monotonicity of (b_n) again, we can say that for any i between k and l, we have

$$
\begin{aligned}
|b_{c_i} - i| &\leq |b_{c_i} - b_{c_k}| + |b_{c_k} - k| + |k - i| && (5.1)\\
&\leq |b_{c_l} - b_{c_k}| + |b_{c_k} - k| + |l - k| && (5.2)\\
&\leq (|l - k| + 1) + 1 + |l - k| && (5.3)\\
&= 2|l - k| + 2 && (5.4)\\
&\leq 2|\lfloor b_{m+1} \rfloor - \lfloor b_m \rfloor| + 2 && (5.5)\\
&\leq 2|b_{m+1} - b_m| + 4 && (5.6)\\
&\leq 2a_{m+1} + 4 && (5.7)\\
&\leq 2C \cdot \frac{b_{m+1}}{m+1} + 4 && (5.8)
\end{aligned}
$$

where the final step holds true for some $C \in \mathbb{R}^+$ (which does not depend on i) because $\mathfrak{b} = \Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$, since $\mathfrak{a}$ is moderate. Now notice that setting $m = c_i - 1$ guarantees $k < i \leq l$, so that we have

$$b_{c_i} - i \leq 2C \cdot \frac{b_{c_i}}{c_i} + 4$$

Now, since we already know that b_{c_i} is bounded below by i by definition, and since (c_i) is monotone and unbounded, we have that

$$b_{c_i} = i + o(b_{c_i})$$

and therefore (b_{c_n}) has growth order $\mathfrak{n}$, and $\mathfrak{b} \circ \mathfrak{c} = \mathfrak{n}$ as desired.

□

5.5. Composition groups

If we find a set of growth orders that is both closed under composition *and* contains an inverse for each of its elements, then it carries the structure of not only a monoid, but a *group*. For instance, consider the set of power growth orders with a positive power, taking the form $\mathfrak{n}^p$ with $p > 0$. We have that $\mathfrak{n}^p \circ \mathfrak{n}^q = \mathfrak{n}^{pq}$ and $\mathrm{inv}(\mathfrak{n}^p) = \mathfrak{n}^{1/p}$, so that this set of growth orders has the same group structure as $\mathbb{R}_\times^{>0}$, the group of positive real numbers under multiplication, which is isomorphic to the group $\mathbb{R}_+$ of real numbers under addition.

We can also consider the group of growth orders taking the form $\mathfrak{n}^p\mathfrak{l}^q$, where $p > 0$ and q is any real number. If we represent elements of this group by tuples (p, q), then the group law of this set of growth orders is given by

$$(p, q) \circ (r, s) = (pr, ps + q)$$

and the formula for the inverse of an element is

$$(p, q)^{-1} = (p^{-1}, -q/p)$$

This group has a normal subgroup, namely the set of all elements of the form $(1, q)$, since $(1, q) \circ (1, s) = (1, q + s)$. This means that the structure of this group can be expressed in the form $\mathbb{R}_{\times}^{>0} \ltimes_{\psi} \mathbb{R}_{+}$ where the homomorphism $\psi : \mathbb{R}_{\times}^{>0} \to \mathrm{Aut}(\mathbb{R}^{+})$ is given by

$$\psi : p \mapsto (s \mapsto ps)$$

6. Closed chains

We have already examined a few different subsets of $\mathcal{S}(\mathbb{R}^+)/\sim$, consisting of growth orders subject to certain "niceness" conditions, such as *moderateness* and *monotonicity*. In order to "zoom in" on a particular subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ and do a deeper analysis there, we would certainly like it so satisfy a few criteria. For instance, $\mathcal{G}$ should be closed under most of the operations that we would like to perform, such as $+, \cdot, \div$, and Σ. It is often troublesome to deal with $\mathcal{S}(\mathbb{R}^+)/\sim$ because it contains many growth orders with erratic and oscillatory behavior that hardly even match our intuition of what a "growth order" should mean, so we would also like $\mathcal{G}$ to exclude many of these pathological sequences. At best, we might even hope for *trichotomy* to hold in $\mathcal{G}$ - that is, for any two growth orders in $\mathcal{G}$ to be comparable, or for $\mathcal{G}$ to be a *chain*.

In this section, we will see how to construct chains $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ that are closed under all of the familiar operations on growth orders.

6.1. Failed attempts

6.2. SR-regularity and closure

Definition 18. *We say that a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ is* ***moderate*** *if every element of $\mathcal{G}$ is moderate.*

Definition 19. *We say that a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ has the* ***monotone quotient property*** *if every quotient of elements of $\mathcal{G}$ is monotone.*

Recall that monotone sequences are always comparable to $\mathbf{1}$, and two sequences are comparable if and only if their quotient is comparable to $\mathbf{1}$. This means that the monotone quotient property guarantees trichotomy in $\mathcal{G}$, and is in fact a much stronger property (as we shall soon see).

Definition 20. *We say that a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ is* ***SR-regular*** *if, for every element $\mathfrak{a} \in \mathcal{G}$, either $\mathfrak{a}$ or $\mathfrak{a}^{-1}$ has an inverse under Σ in $\mathcal{G}$. That is, for any $\mathfrak{a} \in \mathcal{G}$, there exists $\mathfrak{b} \in \mathcal{G}$ such that either $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$.*

Definition 21. *Given a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$, denote by $\Sigma\mathcal{G}$ and $\mathrm{R}\mathcal{G}$ the following sets:*

$$\Sigma\mathcal{G} = \{\Sigma\mathfrak{a} \ : \ \mathfrak{a} \in \mathcal{G}\}$$

$$\mathrm{R}\mathcal{G} = \{\mathfrak{a}^{-1} \ : \ \mathfrak{a} \in \mathcal{G}\}$$

Proposition 54. *If $\mathcal{G}$ is SR-regular, then we have that $\mathcal{G} \subset \Sigma\mathcal{G} \cup \mathrm{R}\Sigma\mathcal{G}$.*

Proof. Follows from the definition of SR-regularity. □

Definition 22. *If $\mathcal{G} \in \mathcal{S}(\mathbb{R}^+)/\sim$ is SR-regular, then by the previous proposition we may recursively define a sequence of nested subsets*

$$\mathcal{G} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_n \subset \cdots$$

*defined by $\mathcal{G}_{n+1} = \Sigma\mathcal{G}_n \cup \mathrm{R}\Sigma\mathcal{G}_n$. Define the **SR-closure** of $\mathcal{G}$ to be the set*

$$\overline{\mathcal{G}} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$$

Proposition 55. *Suppose $\mathcal{G}$ is moderate and SR-regular and satisfies the monotone-quotient property. Then $\overline{\mathcal{G}}$ is also moderate and SR-regular and satisfies the monotone-quotient property, and is furthermore closed under partial sums and reciprocals.*

Proof. The fact that $\overline{\mathcal{G}}$ is moderate follows from the fact that $\Sigma\mathfrak{a}$ and $\mathfrak{a}^{-1}$ are moderate whenever $\mathfrak{a}$ is moderate.

To prove that $\overline{\mathcal{G}}$ has the monotone-quotient property, we will prove inductively that each $\mathcal{G}_n$ has this property, and since the $\mathcal{G}_n$ are nested and $\overline{\mathcal{G}}$ is their union, it must also have this property. Suppose that all quotients of elements of $\mathcal{G}_n$ are monotone. By the definition of $\mathcal{G}_{n+1}$, each element $\mathfrak{a} \in \mathcal{G}_{n+1}$ is either equal to $\Sigma\mathfrak{a}'$ or $(\Sigma\mathfrak{a}')^{-1}$ for some $\mathfrak{a}' \in \mathcal{G}_n$. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}_{n+1}$ be arbitrary. WLOG, there are three cases to consider:

1. $\mathfrak{a} = \Sigma\mathfrak{a}'$ and $\mathfrak{b} = \Sigma\mathfrak{b}'$, with $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$
2. $\mathfrak{a} = \Sigma\mathfrak{a}'$ and $\mathfrak{b} = (\Sigma\mathfrak{b}')^{-1}$, with $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$
3. $\mathfrak{a} = (\Sigma\mathfrak{a}')^{-1}$ and $\mathfrak{b} = (\Sigma\mathfrak{b}')^{-1}$, with $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$

In case 2, we would have $\mathfrak{a}$ containing a monotone increasing sequence and $\mathfrak{b}$ containing a monotone decreasing sequence, so that the quotient $\mathfrak{a}/\mathfrak{b}$ contains a monotone increasing element equal to the elementwise quotient of these two sequences. In case 1, we have that $\mathfrak{a}/\mathfrak{b} = \Sigma\mathfrak{a}'/\Sigma\mathfrak{b}'$ is monotone by 29, since $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$. Similarly, in case 3, we have that $\mathfrak{a}/\mathfrak{b} = \Sigma\mathfrak{b}'/\Sigma\mathfrak{a}'$ is monotone. Thus, any quotient of elements of $\mathcal{G}_{n+1}$ has the monotone quotient property, given that $\mathcal{G}_n$ has this property. Since $\mathcal{G}_0$ has this property by assumption, we have that each $\mathcal{G}_n$ and therefore $\overline{\mathcal{G}}$ has the monotone quotient property.

To see why $\overline{\mathcal{G}}$ is SR-regular, let $\mathfrak{a} \in \overline{\mathcal{G}}$ so that $\mathfrak{a} \in \mathcal{G}_n$ for some $n \in \mathbb{N}$ or $n = 0$. If $n = 0$, we have that $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$ for some $\mathfrak{b} \in \mathcal{G}_0$ because $\mathcal{G}_0$ is SR-regular by assumption. If $n > 0$, then $\mathcal{G}_n = \Sigma\mathcal{G}_{n-1} \cup \mathrm{R}\Sigma\mathcal{G}_{n-1}$, meaning that either $\mathfrak{a} \in \Sigma\mathcal{G}_{n-1}$ or $\mathfrak{a} \in \mathrm{R}\Sigma\mathcal{G}_{n-1}$. In the former case we have $\mathfrak{a} = \Sigma\mathfrak{b}$, and in the latter case we have $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$, where $\mathfrak{b} \in \mathcal{G}_{n-1} \subset \overline{\mathcal{G}}$. In any case, there exists $\mathfrak{b} \in \overline{\mathcal{G}}$ such that $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$, making $\overline{\mathcal{G}}$ SR-regular as claimed.

Finally, we show that $\overline{\mathcal{G}}$ is closed under sums and reciprocals. If $\mathfrak{a} \in \mathcal{G}_n \subset \overline{\mathcal{G}}$, then we have that $\Sigma\mathfrak{a} \in \mathcal{G}_{n+1} \subset \overline{\mathcal{G}}$. Further, we have that either $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$ for some $\mathfrak{b} \in \mathcal{G}_m$. In the former case, we have $\mathfrak{a}^{-1} = (\Sigma\mathfrak{b})^{-1} \in \mathcal{G}_{m+1}$. In the latter case, we have $\mathfrak{a}^{-1} = \Sigma\mathfrak{b} \in \mathcal{G}_{m+1}$. Thus, $\mathfrak{a}^{-1} \in \mathcal{G}_{m+1} \subset \overline{\mathcal{G}}$ in either case, and $\overline{\mathcal{G}}$ is closed under reciprocals. □

Consider the following special set of growth orders:

Definition 23. *Let the set of growth orders*

$$\mathcal{M} = \{\mathfrak{n}^p : p \in \mathbb{Z}\}$$

be called the ***minimal seed set****.*

Clearly the growth orders $\mathfrak{n}^p$ are both monotone and moderate for all $p \in \mathbb{Z}$. Further, this set is SR-regular because

1. $\mathfrak{n}^p = \Sigma\mathfrak{n}^{p-1}$ for $p = 1, 2, \cdots$
2. $\mathfrak{n}^p = \mathrm{R}\Sigma\mathfrak{n}^{-p-1}$ for $p = -1, -2, \cdots$
3. $\mathfrak{n}^p = \Sigma\mathfrak{n}^{-2}$ for $p = 0$

Thus, we may consider the SR-closure $\overline{\mathcal{M}}$. This set of growth orders is noteworthy because it is the *smallest* of all SR-closed sets of growth orders!

Definition 24. *If $\mathcal{G}$ is an SR-regular set of growth orders, then $\overline{\mathcal{M}} \subset \overline{\mathcal{G}}$.*

Proof. Let $\mathfrak{a} \in \overline{\mathcal{G}}$ be arbitrary. Then we have that $\Sigma\mathfrak{a} \geq \mathbf{1}$, meaning that $\Sigma^3\mathfrak{a} \geq \mathfrak{n}^2$. Thus we have that $\mathrm{R}\Sigma^3\mathfrak{a} \leq \mathfrak{n}^{-2}$ and therefore $\Sigma\mathrm{R}\Sigma^3\mathfrak{a} = \mathbf{1}$. Hence, since $\overline{\mathcal{G}}$ is closed under Σ and R, we have that $\mathbf{1} \in \overline{\mathcal{G}}$.

Finally, since $\mathfrak{n}^p = \Sigma^p\mathbf{1}$ and $\mathfrak{n}^{-p} = \mathrm{R}\Sigma^p\mathbf{1}$ for all $p \in \mathbb{N}$, we have that $\overline{\mathcal{G}}$ contains $\mathfrak{n}^p$ for all $p \in \mathbb{Z}$. Thus, it contains $\mathcal{M} = \mathcal{M}_0$ as a subset, and since it is closed under Σ and R, it follows inductively that $\overline{\mathcal{G}}$ contains $\mathcal{M}_n$ for all $n \in \mathbb{N}$ and therefore their union $\overline{\mathcal{M}}$ is also contained in $\overline{\mathcal{G}}$, as claimed. □

Exercise 6 Find a finite SR-regular set $\mathcal{G}$ such that $\overline{\mathcal{G}} = \overline{\mathcal{M}}$.

Exercise 7 What is the smallest possible cardinality that a moderate SR-regular set can have?

6.3. Nested log sums

In this section, we will "get down in the weeds" by taking a close look at one particular SR-closed set, and completely characterizing all of the growth orders contained in it. Specifically, we will determine precisely what growth orders are contained in the SR-closure of the *minimal seed set* $\mathcal{M}$, and thereby determine which growth orders are contained in *all* SR-closed chains of $\mathcal{S}(\mathbb{R}^+)/\sim$. The answer is rather surprising: the growth orders contained in $\overline{\mathcal{M}}$ are precisely those taking the form

$$\mathfrak{l}(p_0, \cdots, p_m) = \mathfrak{n}^{p_0}\mathfrak{l}_1^{p_1} \cdots \mathfrak{l}_m^{p_m}$$

for $p_0, \cdots, p_m \in \mathbb{Z}$. This has a shocking implication - namely, that any set of sequences in which you can take both partial sums and reciprocals contains sequences with the growth order of

$$n^{p_0} \cdot (\log n)^{p_1} \cdot \cdots \cdot (\overbrace{\log \cdots \log}^{m \text{ nested logs}} n)^{p_m}$$

for $p_0, \cdots, p_m \in \mathbb{Z}$. The moral of this story is: if you want to be able to take partial sums and reciprocals of sequences, you cannot avoid dealing with strange growth orders involving nested logarithms!

Before resolving this, however, we must prove several "niceness" properties for growth orders taking the above form. Along the way, we will learn how to compute the growth order of the partial sums of any product of powers of nested logarithms.

Proposition 56. *For all $m \in \mathbb{N}$, the growth order $\mathfrak{l}_m$ is moderate and monotone.*

Proof. Clearly $\mathfrak{l} = \mathfrak{l}_1$ is moderate and monotone. Since the composition of two moderate and monotone growth orders is also moderate and monotone, and $\mathfrak{l}_{m+1} = \mathfrak{l}_m \circ \mathfrak{l}$, the desired result follows by induction. □

Proposition 57. *For all $p \in \mathbb{R}^+$ and $i, j \in \mathbb{N}$ with $i < j$, we have $\mathfrak{l}_j^p \leq \mathfrak{l}_i$. Additionally, $\mathfrak{l}_i^p \leq \mathfrak{n}$ for all $p \in \mathbb{R}^+$.*

Proof. Let $(a_n) \in \mathfrak{l}_i$ be such that $a_n > 1$ for all $n \in \mathbb{N}$, so that the sequence $(\log a_n)$ has growth order $\mathfrak{l}_{i+1}$. Since $\log x \leq x$ for all $x \in \mathbb{R}$, we have that $\log a_n^{1/p} \leq a_n^{1/p}$ for all $n \in \mathbb{N}$ and $p \in \mathbb{R}^+$, or equivalently $(\log a_n)^p \leq p^p a_n$. The LHS of this inequality has growth order $\mathfrak{l}_{i+1}^p$ and the RHS has growth order $\mathfrak{l}_i$, so we have that $\mathfrak{l}_{i+1}^p \le \mathfrak{l}_i$ for all $p \in \mathbb{R}^+$ and $i \in \mathbb{N}$. This means that if $i < j$, we have

$$\mathfrak{l}_j^p \le \mathfrak{l}_{j-1} \le \cdots \le \mathfrak{l}_{i+1} \le \mathfrak{l}_i \le \mathfrak{n}$$

and therefore $\mathfrak{l}_j^p \le \mathfrak{l}_i \le \mathfrak{n}$, as desired. □

Proposition 58. *For all $p_0, \cdots, p_m \in \mathbb{R}$, the growth order $\mathfrak{l}(p_0, \cdots, p_m)$ is monotone.*

Proof. We proceed by induction on m. Suppose that this is true for some $m \in \mathbb{N}$. Since $\mathfrak{l}(q_0, \cdots, q_m)$ is monotone for all $q_0, \cdots, q_m \in \mathbb{R}$, and $\mathfrak{l} > 1$ is also monotone, we have that the growth order

$$\mathfrak{l}(q_0, \cdots, q_m) \circ \mathfrak{l} = \mathfrak{l}(0, q_0, \cdots, q_m)$$

is monotone. Now notice that, since $\mathfrak{l}$ absorbs $\mathfrak{n}^{1/2}$, we also have that $\mathfrak{l}(0, q_0, \cdots, q_m)$ absorbs $\mathfrak{n}^{1/2}$. Further, by the above proposition, we have that $\mathfrak{n}^{1/2} \le \mathfrak{n}\mathfrak{l}(0, q_0, \cdots, q_m)$. Therefore, by 51, we have

$$\Sigma\mathfrak{l}(0, q_0, \cdots, q_m) = \mathfrak{n}\mathfrak{l}(0, q_0, \cdots, q_m) = \mathfrak{l}(1, q_0, \cdots, q_m)$$

which is monotone because it is a partial sum. This further implies that $\mathfrak{l}(1, q_0, \cdots, q_m)^r$, or $\mathfrak{l}(r, rq_0, \cdots, rq_m)$, is monotone, for any $r \in \mathbb{R}$. It follows that $\mathfrak{l}(p_0, \cdots, p_{m+1})$ is monotone for any $p_0, \cdots, p_{m+1} \in \mathbb{R}$, since any tuple $(p_0, \cdots, p_{m+1})$ with $p_0 \neq 0$ can be written in the form $(r, rq_0, \cdots, rq_m)$ for some $r, q_0, \cdots, q_m \in \mathbb{R}$, and the case of $p_0 = 0$ has already been considered. Thus, the inductive step is proven.

The base case of $m = 0$ is clearly true, since $\mathfrak{l}(p_0) = \mathfrak{n}^{p_0}$ is monotone for any $p_0 \in \mathbb{R}$. Hence, by induction, we have that $\mathfrak{l}(p_0, \cdots, p_m)$ is monotone for any $p_0, \cdots, p_m \in \mathbb{R}$. □

Having proven all of the "niceness" properties we need for nested logarithms, we are now ready to start figuring out how to calculate their partial sums. The next proposition proves the following family of asymptotic identities:

$$\sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$$

$$\sum_{k=2}^{n} \frac{1}{k \log k} = \Theta(\log \log n)$$

$$\sum_{k=3}^{n} \frac{1}{k \log k \log \log k} = \Theta(\log \log \log n)$$

$$\cdots$$

Proposition 59. *For all $m \in \mathbb{N}$, we have*

$$\Sigma(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1} = \mathfrak{l}_{m+1}$$

Proof. We shall prove this by induction. Suppose that this claim is true for some $m \in \mathbb{N}$. Notice that

$$P(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1} = \frac{(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}}{\mathfrak{l}_{m+1}} = (\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m \mathfrak{l}_{m+1})^{-1}$$

From the previous proposition, we also have that $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$ is moderate. This means that, by 33, the partial sum of $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$ has the growth order of $\log(1+\lambda)$, where $\lambda \in \mathfrak{l}_{m+1}$. But such a sequence simply has growth order $\mathfrak{l}_{m+2}$, meaning that

$$\Sigma(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m \mathfrak{l}_{m+1})^{-1} = \mathfrak{l}_{m+2}$$

and therefore the inductive step is completed. The base case of $m = 0$ is clearly true, since $\Sigma\mathfrak{n}^{-1} = \mathfrak{l}_1$, and thus the theorem is proven. □

Finally, we are ready to generalize to all products of powers of nested logarithms:

Proposition 60. *Let $p_l, \cdots, p_m \in \mathbb{R}$ with $p_l \neq -1$. Then we have that*

$$\Sigma\mathfrak{l}(-1, \cdots, -1, p_l, p_{l+1}, \cdots, p_m) = \begin{cases} \mathfrak{l}(0, \cdots, 0, p_l + 1, p_{l+1}, \cdots, p_m) & \text{if } p_l > -1 \\ \mathbf{1} & \text{if } p_l < -1 \end{cases}$$

Proof. First of all, suppose that $p_l < -1$. (We shall handle the second case first.) By 57, we have that

$$\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m) < \mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)$$

for any $\epsilon > 0$. By choosing $\epsilon < -(p_l + 1)$, we can ensure that $p_l + \epsilon < -1$. Now, 59 implies that

$$\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon) = \mathfrak{l}(-1, \cdots, -1) \cdot (\Sigma\mathfrak{l}(-1, \cdots, -1))^{p_l + \epsilon}$$

and from here, 34 implies that $\Sigma\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon) = \mathbf{1}$, since $p_l + \epsilon < -1$. Since $\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m)$ is less than $\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)$, its partial sum also converges, and therefore

$$\Sigma\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m) = \mathbf{1}$$

which proves the second case.

Now suppose that $p_l > -1$. By 57, we have that

$$\mathfrak{l}(-1, \cdots, -1, p_l - \epsilon) \leq \mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m) \leq \mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)$$

for any $\epsilon > 0$. Let us choose $\epsilon < p_l + 1$ so that $p_l - \epsilon > -1$. Now, the ratios

$$\frac{\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)}{\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m)}$$

and

$$\frac{\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m)}{\mathfrak{l}(-1, \cdots, -1, p_l - \epsilon)}$$

are monotone by 58. Also, by 34, we have that

$$\Sigma\mathfrak{l}(-1,\cdots,-1,p_l+\epsilon)=\mathfrak{l}(-1,\cdots,-1,p_l+\epsilon+1)$$

and

$$\Sigma\mathfrak{l}(-1,\cdots,-1,p_l-\epsilon)=\mathfrak{l}(-1,\cdots,-1,p_l-\epsilon+1)$$

and therefore

$$\mathrm{P}\mathfrak{l}(-1,\cdots,-1,p_l-\epsilon)=\mathrm{P}\mathfrak{l}(-1,\cdots,-1,p_l+\epsilon)=\mathfrak{l}_l$$

Thus, by applying the typical squeezing argument from 29, we have that

$$\mathrm{P}\mathfrak{l}(-1,\cdots,-1,p_l,\cdots,p_m)=\mathfrak{l}_l$$

and therefore

$$\Sigma\mathfrak{l}(-1,\cdots,-1,p_l,\cdots,p_m)=\mathfrak{l}(-1,\cdots,-1,p_l+1,\cdots,p_m)$$

as claimed, which completes the proof of the first case. □

The notation used in the above proposition might look a bit arcane. Together with the previous proposition, it describes an algorithm to calculate the growth order of the partial sum of the general growth order $\mathfrak{a}=\mathfrak{l}(p_0,\cdots,p_m)$:

1. If $p_0=p_1=\cdots=p_m=-1$, then the growth order of $\Sigma\mathfrak{a}$ is equal to $\mathfrak{l}_{m+1}$.
2. Otherwise, locate the first value of p_i which is not equal to -1. Say that this occurs at $i=l$, so that $p_l\neq-1$ and $p_i=-1$ for all $i<l$.
3. If $p_l<-1$, then $\Sigma\mathfrak{a}$ is the constant growth order $\mathbf{1}$.
4. If $p_l>-1$, then $\Sigma\mathfrak{a}$ is equal to $\mathfrak{l}_{m+1}\mathfrak{a}$.

This gives us, for instance, the following asymptotic formulae, among many others:

$$\sum_{k=3}^{\infty}\frac{1}{k\sqrt{\log k\log\log k}}=\Theta\Big(\sqrt{\frac{\log n}{\log\log n}}\Big)$$

$$\sum_{k=16}^{\infty}\frac{\sqrt{\log\log k}}{k\log k\sqrt[3]{\log\log\log k}}=\Theta\Big(\frac{(\log\log k)^{3/2}}{\sqrt[3]{\log\log\log k}}\Big)$$

Now that we know how to calculate the growth orders of nested log sums in general, we are ready to prove that these are precisely the growth orders that appear in $\overline{\mathcal{M}}$.

Proposition 61. *The elements of $\overline{\mathcal{M}}$ are precisely the growth orders taking the form $\mathfrak{l}(p_0,\cdots,p_m)$, where $p_0,\cdots,p_m\in\mathbb{Z}$.*

Proof. Define a function size on tuples of integers as follows:

$$\text{size}(p_0, \cdots, p_m) = |p_0| + 2|p_1| + \cdots + 2^m |p_m|$$

Notice that this function has the following property: if $p_l > 0$, then

$$\text{size}(0, \cdots, 0, p_l, \cdots, p_m) > \text{size}(-1, \cdots, -1, p_l - 1, \cdots, p_m)$$

This fact will be important for proving our proposition using a modified type of induction on tuples of integers.

Suppose we are given some $\mathfrak{l}(p_0, \cdots, p_m)$, with $p_0, \cdots, p_m \in \mathbb{Z}$ not all equal to zero, which we wish to show is an element of $\overline{\mathcal{M}}$. Suppose that the sequence $p_0, \cdots, p_m$ begins with l zeroes, so that p_l is the first nonzero integer in the sequence, and

$$\mathfrak{l}(p_0, \cdots, p_m) = \mathfrak{l}(0, \cdots, 0, p_l, \cdots, p_m)$$

If $p_l > 0$, then we may write

$$\mathfrak{l}(0, \cdots, 0, p_l, \cdots, p_m) = \Sigma\mathfrak{l}(-1, \cdots, -1, p_l - 1, \cdots, p_m)$$

as per our previously-derived general formula for sums of powers of nested logs. On the other hand, if $p_l < 0$, we have

$$\mathfrak{l}(0, \cdots, 0, p_l, \cdots, p_m) = \mathrm{R}\Sigma\mathfrak{l}(-1, \cdots, -1, -p_l - 1, \cdots, -p_m)$$

However, notice that in the former case,

$$\text{size}(-1, \cdots, -1, p_l - 1, \cdots, p_m) < \text{size}(0, \cdots, 0, p_l, \cdots, p_m)$$

and in the latter case,

$$\text{size}(-1, \cdots, -1, -p_l - 1, \cdots, -p_m) < \text{size}(0, \cdots, 0, p_l, \cdots, p_m)$$

Thus, in either case, we can express $\mathfrak{l}(p_0, \cdots, p_m)$ as the partial sum or the reciprocal of a partial sum of $\mathfrak{l}(q_0, \cdots, q_m)$, where $\text{size}(q_0, \cdots, q_m) < \text{size}(p_0, \cdots, p_m)$.

Since size only takes nonnegative integer values, we have that $\text{size}(q_0, \cdots, q_m) \leq \text{size}(p_0, \cdots, p_m) - 1$. By repeatedly applying the above process, we decrease the size of the tuple of powers by at least 1 at each step, meaning that we must eventually reach a tuple whose size equals zero. The only such tuple is $(0, \cdots, 0)$. This means that $\mathfrak{l}(p_0, \cdots, p_m)$ can be expressed in terms of $\mathfrak{l}(0, \cdots, 0) = \mathbf{1}$ by repeatedly applying the Σ and $\mathrm{R}\Sigma$ operators. However, since $\overline{\mathcal{M}}$ contains $\mathbf{1}$ and is closed under Σ and $\mathrm{R}\Sigma$, it must contain $\mathfrak{l}(p_0, \cdots, p_m)$ for any $p_0, \cdots, p_m \in \mathbb{Z}$.

We have shown that $\overline{\mathcal{M}}$ contains all $\mathfrak{l}(p_0, \cdots, p_m)$ for $p_0, \cdots, p_m \in \mathbb{Z}$, and now we just need to show that it *only* contains growth orders of this form. Clearly $\mathcal{M}$ only contains growth orders of this form, since it consists of power functions $\mathfrak{n}^{p_0} = \mathfrak{l}(p_0)$ with $p_0 \in \mathbb{Z}$. However, 60 shows that if $\mathfrak{a}$ takes the form $\mathfrak{a} = \mathfrak{l}(p_0, \cdots, p_m)$ with $p_0, \cdots, p_m \in \mathbb{Z}$, then $\Sigma\mathfrak{a}$ and $\mathrm{R}\Sigma\mathfrak{a}$ also take this form. It follows inductively that all elements of $\mathcal{M}_i$ take this form for each $i \in \mathbb{N}$, and therefore all elements of $\overline{\mathcal{M}}$ take this form as well. $\square$

This proposition demonstrates that the nested-logarithm growth orders with integer powers are, in some sense, fundamental to the study of partial sums of sequences: $\overline{\mathcal{M}}$ consists of precisely these growth orders, and every SR-closed set contains $\overline{\mathcal{M}}$.

Exercise 8 Show that, for all $m \in \mathbb{N}$, we have that $\mathfrak{l}_m \in \mathcal{M}_i$ for some $i \leq 2^{m+1}$.

Exercise 9 For each of the following "seed sets" $\mathcal{G}$, show that $\mathcal{G}$ is SR-regular, and characterize the growth orders contained in $\overline{\mathcal{G}}$:

- $\mathcal{G} = \{\mathfrak{n}^{1/2}\}$
- $\mathcal{G} = \{\mathfrak{n}^{1/3}, \mathfrak{n}^{2/3}\}$
- $\mathcal{G} = \{\mathfrak{n}^{1/5}, \mathfrak{n}^{4/5}\}$
- $\mathcal{G} = \{\mathfrak{n}^{2/5}, \mathfrak{n}^{3/5}\}$
- $\mathcal{G} = \{\mathfrak{l}^{1/2}, \mathfrak{n}\mathfrak{l}^{1/2}\}$

6.4. Other operations

We've figured out how to construct a set $\overline{\mathcal{G}}$ that is closed under partial sums and reciprocals while still possessing trichotomy (and, in fact, the stronger monotone quotient property). This makes $\mathcal{G}$ a very agreeable setting for doing analysis, but in an ideal microcosm of $\mathcal{S}\mathbb{R}^+/\sim$, these are not all the operations we'd like to be able to perform. We're missing one *significant* operation, namely multiplication. By choosing a suitable $\mathcal{G}$, can we guarantee that $\overline{\mathcal{G}}$ is closed not only under partial sums and reciprocals, but also products?

We will start off slowly by showing that $\mathcal{G}$ is closed under multiplication and division by $\mathfrak{n}$. But first, we need the following lemma:

> **Proposition 62.** *If $\mathcal{G}$ is SR-regular and $\mathfrak{a}/\mathfrak{n}^p$ is monotone for all $\mathfrak{a} \in \mathcal{G}$ and $p \in \mathbb{R}$, then $\mathfrak{a}/\mathfrak{n}^p$ is also monotone for all $\mathfrak{a} \in \overline{\mathcal{G}}$ and $p \in \mathbb{R}$.*

Proof. We complete this proof by induction. Suppose that $\mathfrak{a}/\mathfrak{n}^p$ is monotone for all $\mathfrak{a} \in \mathcal{G}_i$ and all $p \in \mathbb{R}$. Then, by 29, we have that $\Sigma\mathfrak{a}/\Sigma\mathfrak{n}^p$ is monotone. Since $\Sigma\mathfrak{n}^p = \mathfrak{n}^{p+1}$ for all $p > -1$, we have that $\Sigma\mathfrak{a}/\mathfrak{n}^q$ is monotone for all $q > 0$. We also have that $\Sigma\mathfrak{a}/\mathfrak{n}^q$ is monotone for $q \leq 0$ because both $\Sigma\mathfrak{a}$ and $1/\mathfrak{n}^q$ are monotone increasing. Thus, $\Sigma\mathfrak{a}/\mathfrak{n}^q$ is monotone for all $q \in \mathbb{R}$, and additionally $\mathrm{R}\Sigma\mathfrak{a}/\mathfrak{n}^q$ is monotone for all $q \in \mathbb{R}$, since it is the reciprocal of a monotone growth order. Since all elements $\mathfrak{b} \in \mathcal{G}_{i+1}$ take the form $\Sigma\mathfrak{a}$ or $\mathrm{R}\Sigma\mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{G}_i$, we have that $\mathfrak{b}/\mathfrak{n}^p$ is monotone for all $\mathfrak{b} \in \mathcal{G}_{i+1}$ and $p \in \mathbb{R}$.

Since $\mathfrak{a}/\mathfrak{n}^p$ is monotone for all $\mathfrak{a} \in \mathcal{G}_0$ and $p \in \mathbb{R}$ by hypothesis (the base case), we have that $\mathfrak{b}/\mathfrak{n}^p$ is monotone for all $\mathfrak{b} \in \mathcal{G}_i$ for all $i \in \mathbb{N}$, and therefore this holds for all $\mathfrak{b} \in \overline{\mathcal{G}}$ and all $p \in \mathbb{R}$. □

Exercise 10 Strengthen the above result by proving the statement in which $p \in \mathbb{R}$ is replaced by $p \in A$, where $A \subset \mathbb{R}$ is any additive subgroup of $\mathbb{R}$ containing 1.

Exercise 11 Find a growth order $\mathfrak{a}$ such that $\mathfrak{a}/\mathfrak{n}^p$ is monotone for all $p \in \mathbb{Q}$ but not for all $p \in \mathbb{R}$.

Now we are ready to prove the following modest result.

Proposition 63. *Let $\mathcal{G}$ be a moderate and monotone SR-regular set such that $\mathfrak{a}/\mathfrak{n}^p$ is monotone for all $p \in \mathbb{R}$. If $\mathfrak{b} \in \mathcal{G}_i$, then $\mathfrak{n}\mathfrak{b}, \mathfrak{n}^{-1}\mathfrak{b} \in \mathcal{G}_{i+1}$, for all $i \in \mathbb{N}$.*

Proof. Let $\mathfrak{b} \in \mathcal{G}_i$. Without loss of generality, suppose that $\mathfrak{b} \geq \mathbf{1}$ (for if $\mathfrak{b} \leq \mathbf{1}$, we may apply the same line of reasoning to $\mathfrak{b}^{-1} \geq \mathbf{1}$). Then $\mathfrak{b} = \Sigma\mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{G}_{i-1}$. We clearly have that $\Sigma\mathfrak{b} = \Sigma^2\mathfrak{a} = \mathfrak{n}\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{b}$, meaning that $\mathfrak{n}\mathfrak{b} \in \mathcal{G}_{i+1}$. Now we just need to prove that $\mathfrak{b}/\mathfrak{n} \in \mathcal{G}_{i+1}$.

Suppose that $\mathfrak{a} \geq \mathfrak{n}^p$ for some $p \in (-1, 0)$. Then $\mathfrak{a}/\mathfrak{n}^p$ is monotone by hypothesis and the above lemma. Further, since $\mathfrak{a}$ is moderate, we have that $\mathfrak{a} \leq \mathfrak{n}^q$ for some $q \in \mathbb{R}$, and $\mathfrak{a}/\mathfrak{n}^q$ is also monotone. Since $\mathrm{P}\mathfrak{n}^k = \mathfrak{n}^{-1}$ for all $k > -1$, we may deploy the "squeezing" argument outlined in a previous section to argue that $\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{a}$. In this case, we simply have $\mathfrak{a} = \mathfrak{b}/\mathfrak{n}$, and $\mathfrak{a} \in \mathcal{G}_{i-1} \subset \mathcal{G}_{i+1}$.

Now suppose that it is not the case that $\mathfrak{a} \geq \mathfrak{n}^p$ for some $p \in (-1, 0)$. Since $\mathfrak{a}/\mathfrak{n}^p$ is monotone for all $p \in \mathbb{R}$, we have that $\mathfrak{a}$ is comparable to $\mathfrak{n}^p$ for all $p \in \mathbb{R}$, implying that $\mathfrak{a} < \mathfrak{n}^p$ for all $p \in (-1, 0)$. This means that $\Sigma\mathfrak{a} \leq \mathfrak{n}^{p+1}$ and $\mathrm{R}\Sigma\mathfrak{a} \geq \mathfrak{n}^{-p-1}$ for all $p \in (-1, 0)$, so that, taking $p = -1/2$, we have $\mathrm{R}\Sigma\mathfrak{a} \geq \mathfrak{n}^{-1/2}$. Finally, since $\mathrm{P}\mathfrak{n}^k = \mathfrak{n}^{-1}$ for all $k > -1$ and $\mathrm{R}\Sigma\mathfrak{a} \leq \mathfrak{n}^q$ for some $q \in \mathbb{R}$ by moderateness, we may deploy the "squeezing" argument again to deduce that $\Sigma\mathrm{R}\Sigma\mathfrak{a} = \mathfrak{n}\mathrm{R}\Sigma\mathfrak{a} = \mathfrak{n}/\Sigma\mathfrak{a}$. This means that $\mathrm{R}\Sigma\mathrm{R}\Sigma\mathfrak{a} = \Sigma\mathfrak{a}/\mathfrak{n} = \mathfrak{b}/\mathfrak{n}$. Since $\mathfrak{a} \in \mathcal{G}_{i-1}$, we have that $\mathfrak{b}/\mathfrak{n} = (\mathrm{R}\Sigma)^2\mathfrak{a} \in \mathcal{G}_{i+1}$.

In any case, we have that $\mathfrak{n}\mathfrak{b}$ and $\mathfrak{b}/\mathfrak{n}$ are both elements of $\mathcal{G}_{i+1}$, and the proposition is proven. □

6.5. SR-operators in general

Definition 25. *A **SR-operator** on $\mathcal{S}(\mathbb{R}^+)/\sim$ is defined as a composition of finitely many (or zero) Σ and R operators. That is, the identity function on $\mathcal{S}(\mathbb{R}^+)/\sim$ is an SR-operator, and $\Sigma\Phi$ and $\mathrm{R}\Phi$ are SR-operators whenever Φ is an SR-operator.*

Definition 26. *If Φ is an SR-operator, say that it is **even** if it equals a composition and Σ and R containing an even number of instances of R, and **odd** if it equals a composition of Σ and R containing an odd number of instances of R.*

Proposition 64. *If Φ is an SR-operator and $\mathfrak{a}/\mathfrak{b}$ is monotone, then $\Phi\mathfrak{a}/\Phi\mathfrak{b}$ is also monotone.*

Proof. This is a simple proof by induction. Every SR-operator Φ can be written as a string of Σ and R, and can therefore be related to a simpler SR-operator via one of the two equations $\Phi = \Sigma\Phi'$ or $\Phi = \mathrm{R}\Phi'$. Therefore, if we prove that the validity of the proposition for Φ' implies its validity for both $\Sigma\Phi'$ and $\mathrm{R}\Phi'$, we have that it holds in general by induction. But $\Phi'\mathfrak{a}/\Phi'\mathfrak{b}$ being monotone implies that $\Sigma\Phi'\mathfrak{a}/\Sigma\Phi'\mathfrak{b}$ is monotone by 29, and it also implies that $(\Phi'\mathfrak{a}/\Phi'\mathfrak{b})^{-1} = \mathrm{R}\Phi'\mathfrak{a}/\mathrm{R}\Phi'\mathfrak{b}$ is monotone. Thus, the desired result follows by induction. □

Proposition 65. *If Φ is an even SR-operator, then*

$$\mathfrak{a} \le \mathfrak{b} \implies \frac{\mathfrak{a}}{\Phi\mathfrak{a}} \le \frac{\mathfrak{b}}{\Phi\mathfrak{b}}$$

Proof. Notice that if Φ is an even SR-operator, so that it can be written as a string of Σ and R with an even number of occurrences of R, then it can be expressed in terms of a simpler SR-operator Φ' in one of the following 3 ways:

1. $\Phi = \Sigma\Phi'$
2. $\Phi = \Phi'\Sigma$
3. $\Phi = \mathrm{R}\Phi'\mathrm{R}$

Therefore, we can prove our proposition by induction on the length of strings of Σ and R. If we show that the proposition holds for Φ in either of the above 3 cases given that it holds for Φ', then it holds for SR-operators in general by induction.

So suppose that the proposition holds for Φ', so that for any $\mathfrak{a} \le \mathfrak{b}$ with $\mathfrak{a}/\mathfrak{b}$ monotone, we have that

$$\frac{\mathfrak{a}}{\Phi'\mathfrak{a}} \le \frac{\mathfrak{b}}{\Phi'\mathfrak{b}}$$

or, equivalently,

$$\frac{\mathfrak{a}}{\mathfrak{b}} \le \frac{\Phi'\mathfrak{a}}{\Phi'\mathfrak{b}}$$

Further, by the previous proposition, we have that the ratio $\Phi'\mathfrak{a}/\Phi'\mathfrak{b}$ is monotone.

In the first case, if $\Phi = \Sigma\Phi'$, we have that $\mathfrak{a} \le \mathfrak{b}$ implies $\Phi'\mathfrak{a} \le \Phi'\mathfrak{b}$ by assumption, with $\Phi'\mathfrak{a}/\Phi'\mathfrak{b}$ monotone. This means that, by 4.3, we have $\mathrm{P}\Phi'\mathfrak{a} \le \mathrm{P}\Phi'\mathfrak{b}$, which is equivalent to

$$\frac{\Phi'\mathfrak{a}}{\Phi'\mathfrak{b}} \le \frac{\Sigma\Phi'\mathfrak{a}}{\Sigma\Phi'\mathfrak{b}}$$

and therefore

$$\frac{\mathfrak{a}}{\mathfrak{b}} \le \frac{\Sigma\Phi'\mathfrak{a}}{\Sigma\Phi'\mathfrak{b}} = \frac{\Phi\mathfrak{a}}{\Phi\mathfrak{b}}$$

or equivalently

$$\frac{\mathfrak{a}}{\Phi\mathfrak{a}} \leq \frac{\mathfrak{b}}{\Phi\mathfrak{b}}$$

which proves the proposition for Φ when $\Phi = \Sigma\Phi'$, in the first case.

Now consider the second case, in which $\Phi = \Phi'\Sigma$. If $\mathfrak{a} \leq \mathfrak{b}$ with $\mathfrak{a}/\mathfrak{b}$ monotone, then we have that $\Sigma\mathfrak{a} \leq \Sigma\mathfrak{b}$ with $\Sigma\mathfrak{a}/\Sigma\mathfrak{b}$ monotone by 4.3, and further

$$\frac{\mathfrak{a}}{\mathfrak{b}} \leq \frac{\Sigma\mathfrak{a}}{\Sigma\mathfrak{b}}$$

by 4.3. Since the proposition was assumed to be true for Φ', we therefore have that

$$\frac{\Sigma\mathfrak{a}}{\Sigma\mathfrak{b}} \leq \frac{\Phi'(\Sigma\mathfrak{a})}{\Phi'(\Sigma\mathfrak{b})} = \frac{\Phi\mathfrak{a}}{\Phi\mathfrak{b}}$$

and therefore

$$\frac{\mathfrak{a}}{\mathfrak{b}} \leq \frac{\Phi\mathfrak{a}}{\Phi\mathfrak{b}}$$

or equivalently

$$\frac{\mathfrak{a}}{\Phi\mathfrak{a}} \leq \frac{\mathfrak{b}}{\Phi\mathfrak{b}}$$

which proves the proposition for Φ when $\Phi = \Phi'\Sigma$, completing the inductive step for the second case.

Finally, consider the third case, in which $\Phi = \mathrm{R}\Phi'\mathrm{R}$. Assume again that the proposition holds for Φ'. If $\mathfrak{a} \leq \mathfrak{b}$ are such that $\mathfrak{a}/\mathfrak{b}$ is monotone, then we have that $\mathrm{R}\mathfrak{b} \leq \mathrm{R}\mathfrak{a}$, and $\mathrm{R}\mathfrak{b}/\mathrm{R}\mathfrak{a} = (\mathfrak{a}/\mathfrak{b})^{-1}$ is monotone. By assumption, we have that

$$\frac{\mathrm{R}\mathfrak{b}}{\Phi'\mathrm{R}\mathfrak{b}} \leq \frac{\mathrm{R}\mathfrak{a}}{\Phi'\mathrm{R}\mathfrak{a}}$$

which, by inverting both sides of the inequality, is equivalent to

$$\frac{\mathfrak{a}}{\mathrm{R}\Phi'\mathrm{R}\mathfrak{b}} \leq \frac{\mathfrak{b}}{\mathrm{R}\Phi'\mathrm{R}\mathfrak{b}}$$

or

$$\frac{\mathfrak{a}}{\Phi\mathfrak{b}} \leq \frac{\mathfrak{b}}{\Phi\mathfrak{b}}$$

which completes the proof in the third case, in which Φ takes the form $\Phi = \mathrm{R}\Phi'\mathrm{R}$.

Since the base case in which Φ is the identity operator holds trivially, we have that the proposition holds in general by induction on the complexity of strings of Σ and R. □

Additionally, because every odd SR-operator can be written in the form $\mathrm{R}\Phi'$ for some even SR-operator Φ', the above proposition implies the following corollary:

Proposition 66. *If Φ is an odd SR-operator, then*

$$\mathfrak{a} \leq \mathfrak{b} \implies \frac{\mathfrak{b}}{\Phi\mathfrak{b}} \leq \frac{\mathfrak{a}}{\Phi\mathfrak{a}}$$

We may now prove the following proposition, which makes it incredibly easy to compare the outputs of SR-operators:

Proposition 67. *If $\mathfrak{a} = \Phi\Sigma\mathfrak{a}'$ and $\mathfrak{b} = \Phi\mathrm{R}\Sigma\mathfrak{b}'$ for some growth orders $\mathfrak{a}, \mathfrak{a}', \mathfrak{b}, \mathfrak{b}'$ and SR-operator Φ, then $\mathfrak{a}/\mathfrak{b}$ is monotone. Additionally, if Φ is even, then $\mathfrak{a} \geq \Phi\mathbf{1} \geq \mathfrak{b}$, whereas if Φ is odd, then $\mathfrak{a} \leq \Phi\mathbf{1} \leq \mathfrak{b}$.*

Proof. Suppose the growth orders $\mathfrak{a}, \mathfrak{a}', \mathfrak{b}, \mathfrak{b}'$ satisfy

$$\mathfrak{a} = \Phi\Sigma\mathfrak{a}'$$

$$\mathfrak{b} = \Phi\mathrm{R}\Sigma\mathfrak{b}'$$

for some SR-operator Φ. Notice that $\Sigma\mathfrak{a}' \geq \mathbf{1}$ and $\Sigma\mathfrak{a}'/\mathbf{1} = \Sigma\mathfrak{a}'$ is monotone. This implies that $\Phi\Sigma\mathfrak{a}'/\Phi\mathbf{1} = \mathfrak{a}/\Phi\mathbf{1}$ is monotone by 64. Further, we have that $\mathfrak{a} \geq \Phi\mathbf{1}$ if Φ is even and $\mathfrak{a} \leq \Phi\mathbf{1}$ if Φ is odd.

Similarly, we have that $\mathrm{R}\Sigma\mathfrak{b}' \leq \mathbf{1}$ and $\mathrm{R}\Sigma\mathfrak{b}'/\mathbf{1} = \mathrm{R}\Sigma\mathfrak{b}'$ is monotone. This again means that $\Phi\mathrm{R}\Sigma\mathfrak{b}'/\Phi\mathbf{1} = \mathfrak{b}/\Phi\mathbf{1}$ is monotone, and further that $\mathfrak{b} \leq \Phi\mathbf{1}$ if Φ is even and $\mathfrak{b} \geq \Phi\mathbf{1}$ if Φ is odd.

Hence, if Φ is even, then the growth orders $\mathfrak{a}/\Phi\mathbf{1}$ and $\Phi\mathbf{1}/\mathfrak{b}$ are both monotone and $\geq \mathbf{1}$, meaning that their product is monotone and $\geq \mathbf{1}$, so that $(\mathfrak{a}/\Phi\mathbf{1})(\Phi\mathbf{1}/\mathfrak{b}) = \mathfrak{a}/\mathfrak{b}$ is monotone and $\geq \mathbf{1}$, and therefore $\mathfrak{a} \geq \mathfrak{b}$. On the other hand, if Φ is odd, then both $\mathfrak{a}/\Phi\mathbf{1}$ and $\Phi\mathbf{1}/\mathfrak{b}$ are monotone and $\leq \mathbf{1}$, so that their product $\mathfrak{a}/\mathfrak{b}$ is monotone and $\leq \mathbf{1}$, and additionally $\mathfrak{a} \leq \mathfrak{b}$. Thus follows the claimed proposition. □

6.6. Monoid structure

The SR-operators, together with the identity transformation on growth orders I, have the structure of a *monoid* under the binary operation of composition. Since an SR-operator is defined to be a finite composition of Σ and R, we have that this monoid is generated by the operators Σ and R. So far we have only considered the *action* of the SR-operators on $\mathcal{S}(\mathbb{R}^+)/\sim$, but we can also consider the structure of the monoid that they give rise to. In particular, a natural question to ask is whether we can give a concise description of this monoid without making reference to $\mathcal{S}(\mathbb{R}^+)/\sim$, perhaps by finding a *monoid presentation*, i.e. a set of relations between the generators Σ and R from which the entire structure of the monoid can be deduced.

The first step in searching for a presentation for this monoid would be to find nontrivial relations that hold between Σ and R. For instance, the most basic relation is

$$\mathrm{R}^2 = \mathrm{I}$$

which is true because $\mathrm{R}^2\mathfrak{a} = (\mathfrak{a}^{-1})^{-1} = \mathfrak{a}$ for any growth order $\mathfrak{a}$.

This is not the only nontrivial relation between Σ and R. We also have the following relation:

$$\mathrm{R}\Sigma\mathrm{R}\Sigma^3 = \Sigma\mathrm{R}\Sigma^3$$

Why is this the case? If $\mathfrak{a}$ is an arbitrary growth order, we have that $\Sigma\mathfrak{a} \geq \mathbf{1}$, from which we have that $\Sigma^3\mathfrak{a} \geq \mathfrak{n}^2$ and therefore $\mathrm{R}\Sigma^3\mathfrak{a} \leq \mathfrak{n}^{-2}$. Since the partial sums of any sequence with a growth order less than or equal to $\mathfrak{n}^{-2}$ must converge, we have that $\Sigma\mathrm{R}\Sigma^3\mathfrak{a} = \mathbf{1}$ for any growth order $\mathfrak{a}$, and thence follows the above relation. By similar reasoning, we also have the following relations:

$$\Sigma\mathrm{R}\Sigma^3\Sigma = \Sigma\mathrm{R}\Sigma^3$$

$$\Sigma\mathrm{R}\Sigma^3\mathrm{R} = \Sigma\mathrm{R}\Sigma^3$$

As it happens, these are not the only nontrivial relations.

Let us define a sequence of SR-operators Λ_m as follows. Let $\Lambda_0 = \mathrm{I}$ and

$$\Lambda_{m+1} = \Sigma\mathrm{R}\Sigma\Lambda_m\Lambda_{m-1}\cdots\Lambda_0\mathrm{R}$$

for all $m \geq 0$. It can be proven by induction that each operator Λ_m is even, and that

$$\Lambda_m\mathbf{1} = \mathfrak{l}_m$$

for all $m > 0$.

6.7. Exponential extensions

We have seen how to use SR-regularity to construct chains of growth orders with certain favorable properties, such as moderateness, the monotone-quotient property, and SR-closure. Some of them, such as $\overline{\mathcal{M}}$, incidentally have other advantageous properties such as closure under products. Now we will derive a way of "extending" chains with these properties to obtain larger chains containing a large variety of growth orders and sharing the same favorable properties.

In some ways, what we will do is analogous to the idea of a field extension. Given a field such as $\mathbb{Q}$, some equations like $4x^2 - 1 = 0$ will have solutions, while other equations like $x^2 - 2 = 0$ will not have any solutions. We might want to construct a larger field containing $\mathbb{Q}$ but which also contains a solution to the equation $x^2 - 2 = 0$, without disrupting the field structure. This the motivation behind the construction of the field $\mathbb{Q}(\sqrt{2})$. In our case, we will be considering equations of the form

$$\mathrm{P}\mathfrak{x} = \frac{\mathfrak{x}}{\Sigma\mathfrak{x}} = \mathfrak{a}$$

to be solved for $\mathfrak{x}$, where $\mathfrak{a} \in \mathcal{G}$ is some given growth order in a chain $\mathfrak{g}$. If, for instance, $\mathcal{G} = \overline{\mathcal{M}}$, and $\mathfrak{a} = (\mathfrak{n}\mathfrak{l})^{-1}$, then $\mathfrak{x} = \mathfrak{n}^{-1}$ would be a solution to this equation. However, this equation does not have solutions for all values of $\mathfrak{a} \in \overline{M}$, for instance $\mathfrak{a} = \mathfrak{n}^{-1/2}$, $\mathfrak{a} = \mathbf{1}$, $\mathfrak{a} = \mathfrak{n}$, or $\mathfrak{a} = (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$.

A natural question is whether we can find extensions of $\overline{\mathcal{M}}$ in which these equations have solutions, and which share the favorable properties of moderateness, monotone quotients, SR-regularity, and SR-closure (and even closure under products). Clearly this is impossible for some values of $\mathfrak{a}$. For instance, the equation $\mathrm{P}\mathfrak{x} = \mathfrak{n}$ cannot have any solutions in any extension, because this would mean that $\Sigma\mathfrak{x} = \mathfrak{n}\mathfrak{x} > \mathfrak{x}$, which is not the case for any growth order. To consider another example, the equation $\mathrm{P}\mathfrak{x} = \mathfrak{x}$ does have some solutions, such as $\mathfrak{x} = [2^n]$, but none of these solutions are *moderate*, for we have proven that $\Sigma\mathfrak{x} \geq \mathfrak{n}\mathfrak{x}$ for all moderate growth orders $\mathfrak{x}$. (By similar reasoning, $\mathrm{P}\mathfrak{x} = \mathfrak{n}^{-1}$ also cannot have any moderate solutions.) This means that adjoining any solutions of this equation to $\overline{\mathcal{M}}$ would destroy its moderateness property. To continue the analogy with field extensions, this would be like trying to adjoin a solution to equations like $x = x + 1$ to $\mathbb{Q}$ - doing this would necessarily violate the field laws.

But what about the equation $\mathrm{P}\mathfrak{x} = (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$? We cannot rule out the existence of a solution to this equation for any of the above reasons, although we know that no such solution exists in $\overline{\mathcal{M}}$ because we have already completely classified its elements and their partial sums. The theorem 35 allows us to actually construct such a growth order, however:

$$\sum_{k=2}^{n} \frac{e^{\sqrt{\log k}}}{k\sqrt{\log k}} = \Theta\left(e^{\sqrt{\log n}}\right)$$

which tells us that $\Sigma\mathfrak{x} = \mathfrak{x}\mathfrak{n}\sqrt{\mathfrak{l}}$, and therefore $\mathrm{P}\mathfrak{x} = (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$, if $\mathfrak{x}$ is the growth order of $(e^{\sqrt{\log n}}/n\sqrt{\log n})$. Can we find some way of augmenting $\overline{\mathcal{M}}$ so that it contains this new growth order, while preserving all of its desirable properties?

The following proposition guarantees that moderateness would be preserved when adding not only the above growth order, but also any growth order of the form $[e^{\Sigma\beta}]$, where β is any monotone decreasing sequence with "sufficiently fast decay" (e.g. with growth order $(\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$):

Proposition 68. *If $\mathfrak{b} \leq \mathfrak{n}^{-1}$ and $\beta \in \mathfrak{b}$ is monotone decreasing, then $[e^{\Sigma\beta}]$ is a moderate growth order.*

Proof. Let $\beta = (b_n) \in \mathfrak{b}$ be as stated in the proposition, and let $(c_n) = \Sigma e^{\beta}$. Further, let $m, n \in \mathbb{N}$ be such that $n \leq m \leq 2n$. Since $\mathfrak{b} \leq \mathfrak{n}^{-1}$, there exists a constant C such that $b_n \leq C/n$ for all $n \in \mathbb{N}$. Then we have that

$$\sum_{k=n+1}^{m} b_k \leq \sum_{k=n+1}^{2n} \sum_{k=n+1}^{2n} \frac{C}{k} \leq \sum_{k=n+1}^{2n} \frac{C}{n} = C$$

which means that

$$\exp\left(\sum_{k=1}^{n} b_k\right) \leq \exp\left(\sum_{k=1}^{m} b_k\right) \leq e^C \cdot \exp\left(\sum_{k=1}^{n} b_k\right)$$

and therefore we have that $e^{\Sigma\beta}$ is of moderate growth order, as claimed. ∎

Proposition 69. *Suppose $\mathfrak{a}/\mathfrak{b} > \mathbf{1}$ is monotone with $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ arbitrary. Then, for any constant $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n/b_n > M$ for all $n \geq N$.*

Proof. Let $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ be as stated above, and let $(r_n) \in \mathfrak{a}/\mathfrak{b}$ be a monotone sequence. Since (r_n) is monotone and has a growth order strictly greater than $\mathbf{1}$, we have that for any constant > 0, there exists $N \in \mathbb{N}$ such that r_n exceeds the value of that constant for all $n \geq N$. Additionally, since $(r_n) \sim (a_n/b_n)$, we have that $a_n/b_n \geq Cr_n$ for all n, for some $C > 0$. We may therefore let $N \in \mathbb{N}$ be such that $r_n > M/C$ for all $n \geq N$, and consequently $a_n/b_n \geq M$ for all $n \geq N$, as claimed. □

Proposition 70. *Let $\mathcal{G}$ be a monotone, moderate and SR-regular set that is closed under quotients. Further let $\mathfrak{b} \in \mathcal{G}$ be $< \mathfrak{n}^{-1}$ and have the property that there exists no $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$. Then, if $\beta \in \mathfrak{b}$, for all $\mathfrak{a} \in \mathcal{G}$, the ratio $[e^{\Sigma\beta}]/\mathfrak{a}$ is monotone.*

Proof. Let $\mathfrak{a} \in \mathcal{G}$ be arbitrary. Since $\mathcal{G}$ is SR-regular, we have that either $\mathfrak{a} = \Sigma\mathfrak{a}'$ or $\mathfrak{a} = (\Sigma\mathfrak{a}')^{-1}$ for some $\mathfrak{a}' \in \mathcal{G}$. If the latter is true, then we trivially have that the ratio $[e^{\Sigma\mathfrak{b}}]/\mathfrak{a}$ is monotone, since $[e^{\Sigma\mathfrak{b}}]$ is monotone increasing and $1/\mathfrak{a}$ is monotone increasing.

Suppose instead that $\mathfrak{a} = \Sigma\mathfrak{a}'$. Since $\mathcal{G}$ is closed under quotients, we have that $\mathfrak{a}'/\Sigma\mathfrak{a}' = \mathfrak{a}'/\mathfrak{a}$ is in $\mathcal{G}$, and it therefore has a monotone ratio with $\mathfrak{b}$. We will consider two cases: either $\mathfrak{b}/(\mathfrak{a}'/\mathfrak{a})$ is monotone increasing and $> \mathbf{1}$, or it is monotone decreasing and $< \mathbf{1}$, for by assumption it cannot be $= \mathbf{1}$.

Suppose first that $\mathfrak{b}/(\mathfrak{a}'/\mathfrak{a})$ is monotone increasing and $> \mathbf{1}$. Let $\alpha' = (a'_n) \in \mathfrak{a}'$ be an arbitrary sequence of growth order $\mathfrak{a}'$, and let $\alpha = (a_n) = \Sigma\alpha' \in \mathfrak{a}$. By the previous lemma, and by moderateness, for any $M > 0$, the sequence $b_{n+1}/(a'_{n+1}/a_n)$ eventually exceeds M (for all $n \geq N$ with $N \in \mathbb{N}$). Notice, however, that if we fix some $M \geq 1$, we have

$$1 \leq \frac{1 + b_{n+1}}{1 + \frac{a'_{n+1}}{a_n}} \leq \frac{b_{n+1}}{a'_{n+1}/a_n}$$

for all $n \geq N$ by the mediant inequality, meaning that the quotient in the middle of the inequality exceeds 1 for all $n \geq N$. Since $e^h \geq 1 + h$ for all $h \in \mathbb{R}$, we have that

$$\frac{e^{b_{n+1}}}{1 + \frac{a'_{n+1}}{a_n}} \geq 1$$

for all $n \geq N$, or

$$\frac{e^{b_{n+1}}}{\frac{\sum_{k=1}^{n+1} a'_k}{\sum_{k=1}^{n} a'_k}} = \frac{e^{b_{n+1}}}{a_{n+1}/a_n} \geq 1$$

for all $n \geq N$. But this expression is the ratio between consecutive terms of the sequence e^{β}/α. Since this ratio is greater than 1 for all $n \geq N$, it follows that the sequence is monotone increasing for all $n \geq N$, and therefore its growth order $[e^{\beta}]/\mathfrak{a}$ is monotone increasing as desired.

Next consider the case in which $\mathfrak{b}/(\mathfrak{a}'/\mathfrak{a})$ is monotone decreasing, and define α' and α as before. By the previous lemma and by moderateness once more, we have that for any $\epsilon > 0$, the sequence $(b_{n+1} + b_{n+1}^2)/(a'_{n+1}/a_n)$ is eventually less than ϵ for all $n \geq N$, for some $N \in \mathbb{N}$. (We are using the fact that $(b_{n+1} + b_{n+1}^2)$ also has growth order $\mathfrak{b}$.) Using the mediant inequality again, if we fix some positive $\epsilon < 1$, this means that

$$\frac{b_{n+1} + b_{n+1}^2}{a'_{n+1}/a_n} \leq \frac{1 + b_{n+1} + b_{n+1}^2}{1 + \frac{a'_{n+1}}{a_n}} \leq 1$$

for all $n \geq N$. Now, notice that $e^x \leq 1 + x + x^2$ for all sufficiently small x, meaning that since b_{n+1} tends to zero, we have that $e^{b_{n+1}} \leq 1 + b_{n+1} + b_{n+1}^2$ for all $n \geq N'$, for some $N' \in \mathbb{N}$. This means that

$$\frac{e^{b_{n+1}}}{1 + \frac{a'_{n+1}}{a_n}} \leq \frac{1 + b_{n+1} + b_{n+1}^2}{1 + \frac{a'_{n+1}}{a_n}} \leq 1$$

for all $n \geq \max(N, N')$. But this means that

$$\frac{e^{b_{n+1}}}{\frac{\sum_{k=1}^{n+1} a'_k}{\sum_{k=1}^{n} a'_k}} = \frac{e^{b_{n+1}}}{a_{n+1}/a_n} \leq 1$$

and the LHS is the ratio between consecutive terms of the sequence e^{β}/α. Since these ratios are less than 1 for all $n \geq \max(N, N')$, the sequence must be monotone decreasing for all n above this threshold, and therefore the growth order $[e^{\beta}]/\mathfrak{a}$ is monotone decreasing as claimed, completing our proof. □

We have the following corollary of the above proof:

Proposition 71. *Let $\mathcal{G}$ be a monotone, moderate and SR-regular set that is closed under quotients. Further let $\mathfrak{b} \in \mathcal{G}$ be $< \mathfrak{n}^{-1}$ and have the property that there exists no $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$, and let $\beta \in \mathfrak{b}$. For all $\mathfrak{a} \in \mathcal{G}$, either $[e^{\Sigma\beta}]^p/\mathfrak{a}$ is monotone increasing for all $p > 0$, or monotone decreasing for all $p > 0$.*

Proposition 72. *Let $\mathcal{G}$ be a moderate monotone-quotient SR-regular and SR-closed set that is closed under quotients. Let $\mathfrak{b}, \mathfrak{g} \in \mathcal{G}$ and let $\beta \in \mathfrak{b}$ be monotone decreasing such that $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, and such that there exists no growth order $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$. Then if*

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p)$$

we have that

1. *If $\mathfrak{g} = \Sigma\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$, then $\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ if $p > 0$ and $\mathfrak{a} = \mathbf{1}$ if $p < 0$.*
2. *If $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$ and $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$, then $\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ if $p > 0$ and $\mathfrak{a} = \mathbf{1}$ if $p < 0$.*
3. *If $\mathfrak{g} = \Sigma\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$, then $\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$.*
4. *If $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$ and $\mathfrak{g}'/\mathfrak{g}^{-1} > \mathfrak{b}$, then $\mathfrak{a} = \mathbf{1}$.*

Proof. First let us consider case (1) with $p > 0$. From the proofs of the previous propositions, we have that if $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$, then $\mathfrak{g} < [e^{\Sigma\beta}]^q$ for all $q > 0$. This means that, for instance,

$$\mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^{2p}$$

where there is a monotone ratio between any two of these growth orders. However, by 34, we know that

$$\mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^{2p})\mathfrak{b}$$

and therefore, using 29 and a squeezing argument, we have that

$$\mathrm{P}(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{b}$$

and therefore

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

as claimed. If, on the other hand, $p < 0$, then we have that

$$\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2}$$

and the latter growth order has convergent partial sums, by 34, so the former must as well, meaning that $\mathfrak{a} = \mathbf{1}$. Thus follows the claim for the case of $p < 0$.

Now let us consider case (2) - the proof will be very similar. Suppose first that $p > 0$. This time, we have that since $\mathfrak{g}'/\mathfrak{g}^{-1}$, it follows that $\mathfrak{g} > [e^{\Sigma\beta}]^q$ for all $q < 0$. This means that

$$\mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2} \leq \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$$

where any two of these growth orders has a monotone ratio. By 34, we have that

$$\mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2}) = \mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{b}$$

and therefore by the same squeezing argument, we have

$$\mathrm{P}(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{b}$$

and thus

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

as claimed. Once again, if $p < 0$, we have that

$$\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$$

and the latter has convergent sums, meaning that the former does as well, so $\mathfrak{a} = \mathbf{1}$ and the second part of case (2) follows.

Next, we shall prove the claim for case (3). Because $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$, we have by 31 and the fact that $\Sigma\mathfrak{b} > \mathbf{1}$ that $\Sigma(\mathfrak{g}'/\mathfrak{g}) > \Sigma\mathfrak{b}$, or $\mathfrak{l} \circ \mathfrak{g} > \Sigma\mathfrak{b}$. But this means that $\mathfrak{g} > [e^{\Sigma\beta}]^q$ for all $q > 0$. This implies that

$$\Sigma(\mathfrak{g} \cdot \mathfrak{b}) \geq \Sigma([e^{\Sigma\beta}]^q \cdot \mathfrak{b}) = [e^{\Sigma\beta}]^q$$

for all $q > 0$, with a monotone quotient. From this inequality, we have that

$$\frac{\mathfrak{g} \cdot \mathfrak{b}}{\sqrt{\Sigma(\mathfrak{g} \cdot \mathfrak{b})}} \leq \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{g} \cdot \mathfrak{b} \cdot \Sigma(\mathfrak{g} \cdot \mathfrak{b})$$

with monotone quotients. However, the growth orders on the left and the right both have a sigma ratio equal to $\mathfrak{g} \cdot \mathfrak{b}/\Sigma(\mathfrak{g} \cdot \mathfrak{b})$ by 34 and moderateness. Thus, by 29, we have that $\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$ has the same sigma ratio, meaning that

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \Sigma(\mathfrak{g} \cdot \mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$$

completing our proof of case (3).

Finally, we shall turn to case (4). Using the same line of reasoning as case (3), since $\mathfrak{g}'/\mathfrak{g}^{-1} > \mathfrak{b}$, we have again that $\Sigma(\mathfrak{g}'/\mathfrak{g}^{-1}) = \mathfrak{l} \circ \mathfrak{g}^{-1} > \Sigma\mathfrak{b}$, and therefore $\mathfrak{g}^{-1} > [e^{\Sigma\beta}]^q$ for all $q > 0$, or $\mathfrak{g} < [e^{\Sigma\beta}]^q$ for all $q < 0$. This means that

$$\Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) \leq \Sigma(\mathfrak{b} \cdot [e^{\Sigma\beta}]^{\min(-2,p)}) = \mathbf{1}$$

and therefore, since $\mathfrak{a}$ can be no smaller than $\mathbf{1}$, we have that

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathbf{1}$$

as claimed. □

Proposition 73. *Let $\mathcal{G}$ be a moderate monotone-quotient SR-regular and SR-closed set that is closed under quotients. Further let $\mathfrak{b}, \mathfrak{g} \in \mathcal{G}$ and let $\beta \in \mathfrak{b}$ be monotone decreasing such that $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, and such that there exists no growth order $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$. Then, for any given $p \in \mathbb{R}$, there exists a growth order $\mathfrak{a} \in \mathcal{G}$ such that either*

$$\Sigma\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

or

$$(\Sigma\mathfrak{a})^{-1} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

Proof. By the SR-regularity of $\mathcal{G}$, there exists $\mathfrak{g}' \in \mathcal{G}$ such that either $\mathfrak{g} = \Sigma\mathfrak{g}'$ or $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$. If $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$ or $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$ (depending on which is the case), then we have that setting

$$\mathfrak{a} = \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$$

gives $\Sigma\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ for $p > 0$, by the previous proposition. If, on the other hand, $p < 0$, we may let

$$\mathfrak{a} = \mathfrak{g}^{-1} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^{-p}$$

so that we have $(\Sigma\mathfrak{a})^{-1} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ by the previous proposition. Therefore, the theorem holds for the case in which $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$ or $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$, and we need only consider the case in which $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$ or $\mathfrak{g}'/\mathfrak{g}^{-1} > \mathfrak{b}$. (It is not possible for $\mathfrak{g}'/\mathfrak{g}$ to *exactly equal* $\mathfrak{b}$, by hypothesis.)

Consider now the case in which $\mathfrak{g} = \Sigma\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$. We may equivalently write this inequality as $\mathfrak{g}'/\mathfrak{b} > \mathfrak{g}$. Since $\mathfrak{g} \geq \mathbf{1}$, we have that $\mathfrak{g}'/\mathfrak{b} > \mathbf{1}$, and therefore by SR-regularity of $\mathcal{G}$, we have that $\mathfrak{g}'/\mathfrak{b} = \Sigma\mathfrak{g}''$ for some $\mathfrak{g}'' \in \mathcal{G}$. Since $\Sigma\mathfrak{g}'' = \mathfrak{g}'/\mathfrak{b} > \mathfrak{g} = \Sigma\mathfrak{g}'$, we have that $\mathfrak{g}'' > \mathfrak{g}'$ (for the two growth orders must be comparable, and if $\mathfrak{g}'' \leq \mathfrak{g}'$ were true, it would follow that $\Sigma\mathfrak{g}'' \leq \Sigma\mathfrak{g}'$, which is not the case). Now, by 29, we have that $\mathrm{P}\mathfrak{g}'' \geq \mathrm{P}\mathfrak{g}'$, implying that

$$\frac{\mathfrak{g}''}{\mathfrak{g}'/\mathfrak{b}} = \frac{\mathfrak{g}''}{\Sigma\mathfrak{g}''} \geq \frac{\mathfrak{g}'}{\Sigma\mathfrak{g}'} = \frac{\mathfrak{g}'}{\mathfrak{g}} > \mathfrak{b}$$

Therefore, by the previous proposition, we have that if we let

$$\mathfrak{a} = (\mathfrak{g}'/\mathfrak{b}) \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p = \mathfrak{g}' \cdot [e^{\Sigma\beta}]^p$$

it would follow that $\Sigma\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$, proving the proposition for this case.

For the final case in which $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$ and $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$, we may simply consider the growth order $\mathfrak{g}^{-1} \cdot [e^{\Sigma\beta}]^{-p}$ and notice that it falls under the previous case, for which the proposition was just proven. Thus, the growth order $\mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ has a preimage under Σ or $\mathrm{R}\Sigma$ in all cases, and the proposition is proven. □

Proposition 74. *Let $\mathcal{G}$ be a set of growth orders that is*

1. *moderate,*
2. *monotone-quotient,*
3. *SR-regular,*
4. *SR-closed,*
5. *and closed under quotients.*

Let $\mathfrak{b} \in \mathcal{G}$ be a growth order with $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, and let $\beta \in \mathfrak{b}$ be monotone decreasing. Then, if we denote by $\mathcal{G}[\mathrm{e}^{\Sigma\mathfrak{b}}]_A$ the set

$$\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A = \{\mathfrak{g} \cdot [e^{\Sigma\beta}]^p \ : \ \mathfrak{g} \in \mathcal{G},\ p \in A\}$$

where A is an additive subgroup of $\mathbb{R}$, we have that $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ also satisfies properties (1) through (5).

Proof. Let $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ be as described above. The fact that $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ satisfies (1) follows from 68. The fact that it satisfies (2) follow from 71. The fact that it is SR-regular follows from 73.

To see why $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ is SR-closed, notice that an arbitrary element $\mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ can be rewritten in the form $(\mathfrak{g}/\mathfrak{b}) \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$. It follows from 72 that the partial sum of this growth order is equal to either $(\mathfrak{g}/\mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$ or $(\Sigma\mathfrak{g}) \cdot [e^{\Sigma\beta}]^p$ or $\mathbf{1}$. Clearly all three of these growth orders are in $\mathcal{G}[e_A^{\Sigma\mathfrak{b}}]$, since $\mathcal{G}$ is closed under quotients and SR-closed. Thus follows property (4).

Finally, the quotient of two arbitrary elements $\mathfrak{g}_1 \cdot [e^{\Sigma\beta}]^p$ and $\mathfrak{g}_2 \cdot [e^{\Sigma\beta}]^q$ of $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ can be written as

$$\frac{\mathfrak{g}_1 \cdot [e^{\Sigma\beta}]^p}{\mathfrak{g}_2 \cdot [e^{\Sigma\beta}]^q} = (\mathfrak{g}_1/\mathfrak{g}_2) \cdot [e^{\Sigma\beta}]^{p-q}$$

which is an element of $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ because A is an additive subgroup of $\mathbb{R}$, and because $\mathfrak{g}_1/\mathfrak{g}_2 \in \mathcal{G}$ since $\mathcal{G}$ is closed under quotients. Hence, $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ satisfies (5) as well, and the theorem is proven. □

Proposition 75. *Let $\mathcal{G}$ be a moderate, monotone-quotient, SR-regular, and SR-closed set of growth orders that is also closed under quotients. Then there exists a set $\mathcal{G}_{\text{exp}} \supset \mathcal{G}$ that satisfies each of these five properties, but also has the following property: for any $\mathfrak{b} \in \mathcal{G}_{\text{exp}}$ with $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, there exists $\mathfrak{a} \in \mathcal{G}_{\text{exp}}$ such that $\mathrm{P}\mathfrak{a} = \mathfrak{b}$.*

Proof. Let $\mathcal{P} \subset 2^{\mathcal{S}(\mathbb{R}^+)/\sim}$ be defined as the set of moderate, monotone-quotient, SR-regular and SR-closed supersets of $\mathcal{G}$ that are also closed under quotients. Define a partial ordering on $\mathcal{P}$ by letting $\mathcal{G}_1 \leq \mathcal{G}_2$ iff $\mathcal{G}_1 = \mathcal{G}_2$ or if there exists some $\beta \in \mathfrak{b} \in \mathcal{G}_1$ such that $\mathcal{G}_1[e^{\Sigma\beta}]_A \subset \mathcal{G}_2$ for which the exponential extension $\mathcal{G}_1[e^{\Sigma\beta}]_A$ is defined. Notice that for any chain in this partial ordering, taking the union of all elements in that chain results in an element of $\mathcal{P}$ that is greater than or equal to every element of that chain. Thus, every chain of $\mathcal{P}$ has an upper bound.

Now we may apply Zorn's Lemma and conclude that $\mathcal{P}$ necessarily has a maximal element, namely a set $\mathcal{G}_{\text{exp}} \supset \mathcal{G}$ such that *there exists no $\beta \in \mathfrak{b} \in \mathcal{G}_{\text{exp}}$* for which the extension $\mathcal{G}_{\text{exp}}[e^{\Sigma\mathfrak{b}}]_A$ is defined, for if it were defined, it would be a proper exponential extension of $\mathcal{G}_{\text{exp}}$ satisfying each of the five desired properties. Hence, if $\mathfrak{b} \in \mathcal{G}_{\text{exp}}$ is such that $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, it cannot be the case that there is no $\mathfrak{a} \in \mathcal{G}_{\text{exp}}$ with $\mathrm{P} = \mathfrak{b}$, for if no such $\mathfrak{a}$ existed, the extension $\mathcal{G}_{\text{exp}}[e^{\Sigma\mathfrak{b}}]_A$ could be constructed. Hence, for every such $\mathfrak{b} \in \mathcal{G}_{\text{exp}}$, there must exist $\mathfrak{a} \in \mathcal{G}_{\text{exp}}$ such that $\mathrm{P}\mathfrak{a} = \mathfrak{b}$. □

THE BELOW PROPOSITION IS AN OLD OBSOLETE VERSION THAT I'M KEEPING IN CASE I NEED TO REUSE SOME PARTS:

Proposition 76. *Let $\mathcal{G}$ be a moderate monotone-quotient SR-regular set that is closed under products and logarithms. Let $\mathfrak{b}, \Sigma\mathfrak{b}, \mathfrak{g} \in \mathcal{G}$ and let $\beta \in \mathfrak{b} < \mathbf{1}$ be monotone decreasing such that there exists no growth order $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{l} \circ \mathfrak{c} = \Sigma\mathfrak{b}$.*
Then if

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p)$$

we have that

1. *If $\mathfrak{g} \geq \mathbf{1}$ and $\mathfrak{l} \circ \mathfrak{g} < \Sigma\mathfrak{b}$, then $\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ if $p > 0$ and $\mathfrak{a} = \mathbf{1}$ if $p < 0$.*
2. *If $\mathfrak{g} < \mathbf{1}$ and $\mathfrak{l} \circ \mathfrak{g}^{-1} < \Sigma\mathfrak{b}$, then $\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ if $p > 0$ and $\mathfrak{a} = \mathbf{1}$ if $p < 0$.*
3. *If $\mathfrak{g} \geq \mathbf{1}$ and $\mathfrak{l} \circ \mathfrak{g} > \Sigma\mathfrak{b}$, then $\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$.*
4. *If $\mathfrak{g} < \mathbf{1}$ and $\mathfrak{l} \circ \mathfrak{g}^{-1} > \Sigma\mathfrak{b}$, then $\mathfrak{a} = \mathbf{1}$.*

Proof. First, suppose that $\mathfrak{g} \geq \mathbf{1}$ and $\mathfrak{l} \circ \mathfrak{g} < \Sigma\mathfrak{b}$, and that $p > 0$. Then we have that

$$\mathfrak{l} \circ (\mathfrak{g} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{l} \circ \mathfrak{g} + \mathfrak{l} \circ [e^{\Sigma\beta}]^p = \mathfrak{l} \circ \mathfrak{g} + \Sigma\mathfrak{b} = \Sigma\mathfrak{b}$$

Now, if we let $(g_n) \in \mathfrak{g}$ be a monotone increasing sequence which is always greater than 1, we have that $(\log g_n)$ is also a monotone increasing sequence of positive reals. It must therefore be equal to $\Sigma(g'_n)$ for some other sequence (g'_n) of positive reals. Further, we must have that g'_n/b_n tends to zero as $n \to \infty$, for otherwise we would have $\mathfrak{l} \circ \mathfrak{g} \geq= \Sigma[(g'_n)] \geq \Sigma\mathfrak{b}$, contradicting the assumption that $\mathfrak{l} \circ \mathfrak{g} < \Sigma\mathfrak{b}$. Now, this means that

$$\mathfrak{g} \cdot [e^{\Sigma\beta}]^p = [e^{\Sigma((g'_n)+p\beta)}]$$

where the sequence $(g'_n) + \beta$ tends to zero. Note also that $\Sigma((g'_n) + p\beta) \in \Sigma\mathfrak{b}$. Finally, by 35, we have that

$$\Sigma\big([(g'_n) + p\beta] \cdot [e^{\Sigma((g'_n)+p\beta)}]\big) = [e^{\Sigma((g'_n)+p\beta)}]$$

Now, notice that since g'_n/b_n tends to zero, we have that $(g'_n)+p\beta \sim \beta$, and therefore $[(g'_n)+p\beta] = \mathfrak{b}$. Additionally, recall that $[e^{\Sigma((g'_n)+p\beta)}] = [e^{\Sigma(g'_n)}] \cdot [e^{\Sigma p\beta}] = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$. Therefore, the above equality becomes

$$\mathfrak{a} = \Sigma(\mathfrak{b} \cdot \mathfrak{g} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

as claimed.

Now suppose that $p < 0$. Because $\mathfrak{l} \circ \mathfrak{g} < \Sigma\mathfrak{b}$, we have that $\mathfrak{g} < [e^{\Sigma\mathfrak{b}}]^q$ for all $q > 0$. By choosing $q = -p/2$, we may conclude that

$$\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2}$$

Now, notice that, by the previous part, we have that this upper bound is equal to

$$\frac{\mathfrak{b}[e^{\Sigma\beta}]^{-p/2}}{\left(\Sigma(\mathfrak{b}[e^{\Sigma\beta}]^{-p/2})\right)^2}$$

which, by 34, has convergent partial sums. Therefore, we have that

$$\Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathbf{1}$$

when $p < 0$, completing our proof of the first case. □

6.8. Isomorphic chains

7. More growth order operations

This chapter is meant as a place to list several other miscellaneous interesting operations on growth orders that I've considered.

7.1. Linear transformations

Several of the nice properties of the summation operation Σ explored in previous sections relied solely on the linearity of $\Sigma\alpha$ as a function of the sequence α. In general, if we consider $\mathcal{S}(\mathbb{R}^+)$ as a subset (not a subspace) of the vector space $\mathcal{S}(\mathbb{R})$ and look at other linear transformations on $\mathcal{S}(\mathbb{R})$ under which $\mathcal{S}(\mathbb{R}^+)$ is closed, we can derive several other interesting transformations on growth orders. Let's begin by proving the following general result, which entails that any such transformation can be extended to a well-defined transformation on growth orders:

Proposition 77. *Let* $\mathrm{T} : X \to X$ *be a linear transformation on some subspace* $X \subset \mathcal{S}(\mathbb{R})$*, and let* $X^+ = X \cap \mathcal{S}(\mathbb{R}^+)$*. If* X^+ *is closed under* T*, then* $\alpha \sim \beta \implies \mathrm{T}\alpha \sim \mathrm{T}\beta$ *for all* $\alpha, \beta \in X^+$*.*

Proof. Let $\mathrm{T}X \to X$ be a linear transformation which satisfies the above hypotheses. Then if $[\alpha] \leq [\beta]$ for some sequences $\alpha, \beta \in X^+ \subset \mathcal{S}(\mathbb{R}^+)$, we have by definition that there exists a constant $C > 0$ such that all entries of α are less than all entries of $C\beta$. In other words, the sequence $C\beta - \alpha \in X$ has only positive entries, so that $C\beta - \alpha \in X^+$. This means that $\mathrm{T}(C\beta - \alpha) \in X^+$ since X^+ is closed under T, and further $C\mathrm{T}\beta - \mathrm{T}\alpha \in X^+$ by linearity of T. This means that all entries of the sequence $C\mathrm{T}\beta - \mathrm{T}\alpha$ are positive, so that all entries of $\mathrm{T}\alpha$ are less than all entries of $C\mathrm{T}\beta$. This in turn implies that $[\mathrm{T}\alpha] \leq [\mathrm{T}\beta]$.

Hence, we have shown that $[\alpha] \leq [\beta] \implies [\mathrm{T}\alpha] \leq [\mathrm{T}\beta]$ for all $\alpha, \beta \in X^+$. Therefore, if $\alpha \sim \beta$, we have that both $[\alpha] \leq [\beta]$ and $[\alpha] \geq [\beta]$, implying that both $[\mathrm{T}\alpha] \leq [\mathrm{T}\beta]$ and $[\mathrm{T}\alpha] \geq [\mathrm{T}\beta]$, and therefore $\mathrm{T}\alpha \sim \mathrm{T}\beta$. Hence, $\alpha \sim \beta \implies \mathrm{T}\alpha \sim \mathrm{T}\beta$ for all $\alpha, \beta \in X^+$, as claimed. □

The above proposition shows that certain classes of linear transformations, when applied to sequences with positive elements, produce sequences whose growth order depends only on the growth order of the input. This justifies the following definition:

> **Definition 27.** *If* $\mathrm{T} : X \to X$ *is a linear transformation on a subspace* $X \subset \mathcal{S}(\mathbb{R}^+)$ *such that* $X \cap \mathcal{S}(\mathbb{R}^+)$ *is closed under* T*, then we may define* T *as a transformation on growth orders by letting* $\mathrm{T}\mathfrak{a} = \mathrm{T}[\alpha] = [\mathrm{T}\alpha]$ *for any* $\mathfrak{a} \in \mathcal{S}(\mathbb{R}^+)/\sim$ *and* $\alpha \in \mathfrak{a}$*.*

Let's look at a couple examples of transformations that can be defined in this way. Sometimes it is interesting to consider the *generating functions* of sequences of integers or real numbers $\alpha = (a_n)$, which can be defined as follows:

$$f_\alpha(x) = \sum_{n=1}^{\infty} a_n x^n$$

Note that this sum has potential convergence issues. However, if we restrict x to lie in the interval $x \in (-1, 1)$ and impose additional conditions on the sequence (a_n), we can ensure convergence. For instance, we could consider the subspace of $\mathcal{S}(\mathbb{R})$ consisting of sequences with sub-exponential growth, i.e. the sequences (a_n) such that for each $b > 1$, there exists a constant $C > 0$ such that $|a_n| \leq Cb^n$ for all $n \in \mathbb{N}$. Given this condition, the above sum converges (absolutely, in fact) for all $x \in (-1, 1)$, and it is an easy exercise to verify that this is the case.

For many sequences α, the function $f_\alpha(x)$ has a pole at $x = 1$. Therefore, given a sequence α with at most polynomial growth, it might be interesting to analyze the behavior of the function f_α near the point $x = 1$. One way of doing this would be to consider the sequence of points

$$f_\alpha\left(\frac{1}{2}\right),\ f_\alpha\left(\frac{2}{3}\right),\ f_\alpha\left(\frac{3}{4}\right), \cdots,\ f_\alpha\left(1 - \frac{1}{n}\right), \cdots$$

approaching $x = 1$ gradually from the left. The mapping from the sequence α to the above sequence of values of its generating function actually defines a linear transformation $X \to X$, where $X \subset \mathcal{S}(\mathbb{R})$ is the subspace consisting of all sub-exponential sequences! Let us denote this transformation by $\mathrm{G} : X \to X$. It is easy to verify that $X^+ = X \cap \mathcal{S}(\mathbb{R}^+)$ is closed under G, for the infinite sum

$$f_\alpha(x) = \sum_{n=1}^{\infty} a_n x^n$$

is positive for all $x \in (0, 1)$ and sequences of positive numbers (a_n) for which it converges. We may therefore extend G to a transformation on growth orders, as per the previous definition.

We are able to deduce some special values of G right off the bat. For one, the series identity

$$\sum_{n=1}^{\infty} \binom{p+n}{n} x^n = \frac{x}{(1-x)^{p+1}}$$

for powers $p \in \mathbb{N}$ tells us that $\mathrm{G}\mathfrak{n}^p = \mathfrak{n}^{p+1}$. Additionally, the Taylor series for the logarithm

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

gives us the value $\mathrm{G}\mathfrak{n}^{-1} = \mathfrak{l}$. By checking other low-hanging fruit, we might notice that $\mathrm{G}\mathfrak{a}$ seems to match the value of $\Sigma\mathfrak{a}$ for many sequences $\mathfrak{a}$. This is no coincidence, as the following proposition demonstrates.

Proposition 78. *For all moderate growth orders* $\mathfrak{a}$*, we have* $\mathrm{G}\mathfrak{a} = \Sigma\mathfrak{a}$.

Proof. The following sequence (g_n) has growth order $\mathrm{G}\mathfrak{a}$, by our definition of G:

$$g_n = \sum_{k=1}^{\infty} a_k \left(1 - \frac{1}{n}\right)^k$$

Now define the sequence (c_n) as follows:

$$c_n = \left\lfloor \frac{\ln(1/2)}{\ln(1 - \frac{1}{n})} \right\rfloor$$

so that, for all $n \in \mathbb{N}$, we have

$$\left(1 - \frac{1}{n}\right)^{c_n} \geq \frac{1}{2}$$

and

$$\left(1 - \frac{1}{n}\right)^{c_n+1} \leq \frac{1}{2}$$

Additionally, because $-\ln(1-h) = \Theta(h)$ as $h \to 0$, we have that (c_n) has growth order $\mathfrak{n}$.

Now we will bound the difference between (g_n) and a sequence which has growth order $\Sigma\mathfrak{a}$ in order to show that (g_n) has precisely the growth order $\Sigma\mathfrak{a}$. Consider the tail-end of the series defining g_n:

$$g_n - \sum_{k=1}^{c_n} a_k \left(1 - \frac{1}{n}\right)^k = \sum_{k=c_n+1}^{\infty} a_k \left(1 - \frac{1}{n}\right)^k$$

Let us split up this series as follows (which is allowed because of its absolute convergence):

$$\sum_{j=1}^{\infty} \sum_{k=(c_n+1)2^j}^{(c_n+1)2^{j+1}-1} a_k \left(1 - \frac{1}{n}\right)^k$$

Now, because (a_n) is a moderate sequence, there exists a constant $C > 0$ such that $a_m \leq Ca_n$ for all $n \leq m \leq 2n$. This implies that $a_k \leq C^{j+1} a_{c_n+1}$ for all k between $(c_n+1)2^j$ and $(c_n+1)2^{j+1}-1$, for all $j \in \mathbb{N}$. Additionally, from a previous inequality, we have that $(1-1/n)^k \leq 1/2^{2^j}$ for all $k, j \in \mathbb{N}$ with $k \geq (c_n+1)2^j$. Hence, the above series is bounded above by

$$\sum_{j=1}^{\infty} \sum_{k=(c_n+1)2^j}^{(c_n+1)2^{j+1}-1} C^{j+1} a_{c_n+1} \cdot \frac{1}{2^{2^j}}$$

which can be simplified as follows:

$$a_{c_n+1} \sum_{j=1}^{\infty} \frac{2^j (c_n+1) \cdot C^{j+1}}{2^{2^j}}$$

so that we have the inequality

$$g_n - \sum_{k=1}^{c_n} a_k \left(1 - \frac{1}{n}\right)^k \leq a_{c_n+1}(c_n+1) \sum_{j=1}^{\infty} \frac{2^j C^{j+1}}{2^{2^j}}$$

The LHS, as a sequence of n, has the same growth order as $a_{c_n+1}(c_n+1)$, since the infinite series is just a constant factor. Note that the infinite series converges no matter the value of C because of the superexponential term 2^{2^j} in the denominator of the summand. Now, notice that $a_{c_n+1}(c_n+1)$ has a growth order of $(\mathfrak{a} \circ \mathfrak{c}) \cdot \mathfrak{c}$ by the moderateness of $\mathfrak{a}$. This means that

$$g_n = \sum_{k=1}^{c_n} a_k \left(1 - \frac{1}{n}\right)^k + O(na_n)$$

The sum on the RHS can easily be seen to have growth order $\Sigma\mathfrak{a}$ (because each of the coefficients $(1 - 1/n)^k$ is between 1 and $1/2$), and therefore the above formula asserts that $\mathrm{G}\mathfrak{a} = \Sigma\mathfrak{a} + \mathfrak{b}$ where $\mathfrak{b} \leq \mathfrak{n}\mathfrak{a}$. However, recall that $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$ for all moderate growth orders $\mathfrak{a}$, implying that $\mathfrak{b} \leq \Sigma\mathfrak{a}$ and therefore $\Sigma\mathfrak{a} + \mathfrak{b} = \Sigma\mathfrak{a}$. Thus, we have that $\mathrm{G}\mathfrak{a} = \Sigma\mathfrak{a}$ as claimed. □

Question 5 Is $\mathrm{G}\mathfrak{a} = \Sigma\mathfrak{a}$ for immoderate sub-exponential growth orders $\mathfrak{a}$ as well? What are some specific examples, for instance, what is $\mathrm{G}\mathfrak{a}$ when $\mathfrak{a} = [(e^{\sqrt{n}})]$?

We can also do a similar construction involving *Dirichlet series*. Given a sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R})$, its Dirichlet series is defined as

$$\mathscr{D}_\alpha(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Again, this may or may not converge, depending on the sequence α. However, if we restrict α to the subspace $X \subset \mathcal{S}(\mathbb{R})$ consisting of all sequences with sub-polynomial growth, that is, the sequences with the property that for all $p > 0$, there exists $C > 0$ such that $|a_n| \leq Cn^p$ for all $n \in \mathbb{N}$, then the above series is guaranteed to converge absolutely for all $s > 1$. (This is another easy exercise).

However, as s approaches 1 from above, this function $\mathscr{D}_\alpha(s)$ often approaches infinity, as the terms of the series decay more and more slowly as s grows smaller. In similar fashion to our treatment of generating functions, we might investigate this pole by considering a sequence of values

$$\mathscr{D}_\alpha\left(\frac{3}{2}\right), \ \mathscr{D}_\alpha\left(\frac{4}{3}\right), \ \mathscr{D}_\alpha\left(\frac{5}{4}\right), \ \cdots, \mathscr{D}_\alpha\left(1 + \frac{1}{n}\right), \ \cdots$$

Since $\mathscr{D}_\alpha(s)$ is a linear transformation $\mathcal{S}(\mathbb{R}) \supset X \to \mathbb{R}$ for any fixed value of $s \in (1, \infty)$, and furthermore $\mathscr{D}_\alpha(s) > 0$ when $\alpha \in \mathcal{S}(\mathbb{R}^+)$, we have that the above sequence of values defines a linear transformation $\mathrm{D} : X \to X$ under which $X^+ = X \cap \mathcal{S}(\mathbb{R}^+)$ is closed. We may furthermore define D as a transformation on sub-polynomial growth orders.

Exercise 12 Calculate the following special values of D:

- $\mathrm{D}\mathbb{1} = \mathfrak{n}$
- $\mathrm{D}\mathfrak{l} = \mathfrak{n}^2$
- $\mathrm{D}\mathfrak{l}^2 = \mathfrak{n}^3$

Can you generalize?

We can also define linear transformations such as the following:

Definition 28. *Given a sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$, define its **halving sum** $\mathrm{H}\alpha$ to be the sequence (b_n), where*

$$b_n = a_n + a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor} + \cdots + a_1 = \sum_{k=0}^{\lfloor \log_2 n \rfloor} a_{\lfloor n/2^k \rfloor}$$

and if $\mathfrak{a} = [\alpha]$ is the growth order of α, define $\mathrm{H}\mathfrak{a} = \mathrm{H}[\alpha] = [\mathrm{H}\alpha]$.

As we shall see in the next section, this transformation turns out to be incredibly useful for determining the asymptotic behavior of recurrence relations. The transformation H has a couple of convenient similarities to Σ that make it easier to calculate its values given what we already know about Σ. For instance, it satisfies the following analogue of 29, which allows us to deploy similar "squeezing" arguments to deduce many of its values:

Proposition 79. *If $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a}/\mathfrak{b}$ is monotone, then*

$$\frac{\mathfrak{a}}{\mathrm{H}\mathfrak{a}} \leq \frac{\mathfrak{b}}{\mathrm{H}\mathfrak{b}}$$

Proof. Let $\mathfrak{a} \leq \mathfrak{b}$ with $\mathfrak{a}/\mathfrak{b}$ monotone, so that we may find $\alpha \in \mathfrak{a}, \beta \in \mathfrak{b}, \gamma \in \mathfrak{b}/\mathfrak{a}$ so that γ is a monotone increasing sequence. From the fact that $\gamma = (c_n)$ is monotone increasing, it follows directly that

$$\sum_{k=0}^{\lfloor \log_2 n \rfloor} c_{\lfloor n/2^k \rfloor} a_{\lfloor n/2^k \rfloor} \leq c_n \sum_{k=0}^{\lfloor \log_2 n \rfloor} a_{\lfloor n/2^k \rfloor}$$

which tells us that $[\mathrm{H}(\gamma\alpha)] \leq [\gamma \mathrm{H}\alpha]$. This inequality, however, is equivalent to $[\gamma\alpha/\mathrm{H}(\gamma\alpha)] \geq [\alpha/\mathrm{H}\alpha]$, or $[\beta/\mathrm{H}\beta] \geq [\alpha/\mathrm{H}\alpha]$, which implies that

$$\frac{\mathfrak{a}}{\mathrm{H}\mathfrak{a}} \leq \frac{\mathfrak{b}}{\mathrm{H}\mathfrak{b}}$$

as claimed. □

As it happens, it is quite easy to calculate $\mathrm{H}\mathfrak{a}$ for polynomial growth orders $\mathfrak{a} = \mathfrak{n}^p$. In particular, if $\beta = \mathrm{H}\alpha$ with $\alpha = (n^p)$, we have the upper bound

$$b_n = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor^p \le \sum_{k=0}^{\infty} \left(\frac{n}{2^k} \right)^p = \frac{n^p}{1 - 2^{-p}}$$

and the lower bound

$$b_n = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor^p \ge n^p$$

so that we have both upper and lower bounds of order $\mathfrak{n}^p$, implying that $\mathrm{H}\mathfrak{n}^p = \mathfrak{n}^p$ for all $p > 0$. By the squeezing argument, this implies that $\mathrm{H}\mathfrak{a} = \mathfrak{a}$ for all $\mathfrak{a}$ such that $\mathfrak{n}^p \le \mathfrak{a} \le \mathfrak{n}^q$ with $\mathfrak{a}/\mathfrak{n}^p$ and $\mathfrak{a}/\mathfrak{n}^q$ both monotone for some $p, q > 0$. For instance, this gives us formulae like $\mathrm{H}(\sqrt{\mathfrak{n}}\mathfrak{l}/\mathfrak{l}_2) = \sqrt{\mathfrak{n}}\mathfrak{l}/\mathfrak{l}_2$ for free.

The following proposition facilitates the calculation of $\mathrm{H}\mathfrak{a}$ for growth orders $\mathfrak{a}$ which have logarithmic growth or slower:

Proposition 80. *For all moderate growth orders $\mathfrak{a}$, the following formula holds:*

$$\mathrm{H}(\mathfrak{a} \circ \mathfrak{l}) = (\Sigma\mathfrak{a}) \circ \mathfrak{l}$$

Proof. Consider the sequence $\lambda = (\lceil \log_2(n+1) \rceil)$ of growth order $[\lambda] = \mathfrak{l}$, and let $\mathfrak{a}$ be a moderate growth order with $\alpha = (a_n) \in \mathfrak{a}$. The sequence (b_n) has growth order $\mathrm{H}(\mathfrak{a} \circ \mathfrak{l})$, where:

$$b_n = \sum_{k=0}^{\lfloor \log_2 n \rfloor} a_{\lceil \log_2(\lfloor n/2^k \rfloor + 1) \rceil}$$

Now, notice that for all $n, k \in \mathbb{N}$ with $k \le \lfloor \log_2 n \rfloor$, the following equality holds:

$$\lceil \log_2(\lfloor n/2^k \rfloor + 1) \rceil = \lceil \log_2(n+1) \rceil - k$$

This can be proven by noticing that $\lceil \log_2(n+1) \rceil$ equals $p+1$ precisely when $n \in \{2^p, \cdots, 2^{p+1}-1\}$, and when n is in this set, then $\lfloor n/2^k \rfloor$ is necessarily in the set $\{2^{p-k}, \cdots, 2^{p-k+1} - 1\}$. This means that the above formula for b_n is equivalent to

$$b_n = \sum_{k=0}^{\lfloor \log_2 n \rfloor} a_{\lceil \log_2(n+1) \rceil - k}$$

We may reindex this sum, by reversing the order of summation, although the exact reindexing depends on whether n is a power of two, because this determines whether or not $\lfloor \log_2 n \rfloor + 1 = \lceil \log_2(n+1) \rceil$. When n is a power of two, this is equal to

$$b_n = \sum_{k=2}^{\lfloor \log_2 n \rfloor + 2} a_k$$

whereas when n is not a power of two, it equals

$$b_n = \sum_{k=1}^{\lfloor \log_2 n \rfloor + 1} a_k$$

Since $\mathfrak{a}$ is moderate, both expressions have a growth order of $(\Sigma\mathfrak{a}) \circ \mathfrak{l}$, which gives us the desired equality $\mathrm{H}(\mathfrak{a} \circ \mathfrak{l}) = (\Sigma\mathfrak{a}) \circ \mathfrak{c}$. $\square$

This formula gives us special values like $\mathrm{H}\mathfrak{l}^{-1} = \mathfrak{l}_2$ and $\mathrm{H}(\mathfrak{l}^{-1}\mathfrak{l}_2^{-1}) = \mathfrak{l}_3$. Additionally, since $H\mathfrak{l}^{-2} = \mathbf{1}$ by the above formula, and since $\mathrm{H}\mathfrak{a} \geq \mathbf{1}$ for all growth orders $\mathfrak{a}$ clearly from its definition, we can use the squeezing argument to deduce values of H at even faster-decaying growth orders like $\mathrm{H}\mathfrak{n}^p = \mathbf{1}$ for all $p < 0$.

7.2. Recurrences

Of particular interest in computer science is the following type of recurrence, which is often referred to as a "divide and conquer" recurrence:

$$T(n) = aT(\lfloor n/b \rfloor) + f(n)$$

In the theory of algorithm complexity analysis, *The Master Theorem* is a theorem that determines the growth order of $(T(n))$ in terms of a, b, and the growth order of $(f(n))$, for certain growth orders f. We will shortly prove a result that generalizes the Master Theorem to a slightly broader class of functions $f(n)$.

Proposition 81. *If $f_1, f_2 : \mathbb{N} \to \mathbb{R}$ are such that $[f_1(n)] = [f_2(n)]$, and $T_1, T_2 : \mathbb{N} \to \mathbb{R}^+$ satisfy the recurrences*

$$T_1(n) = aT_1(\lfloor n/b \rfloor) + f_1(n)$$

$$T_2(n) = aT_2(\lfloor n/b \rfloor) + f_2(n)$$

for all sufficiently large $n \in \mathbb{N}$ and for some constants $a > 0, b > 1$, then it follows that $[T_1(n)] = [T_2(n)]$. Additionally, if $[f_1(n)] \leq [f_2(n)]$, then $[T_1(n)] \leq [T_2(n)]$.

Proof. Suppose that T_1, T_2 satsify the above recurrences for all $n \geq N$, where $N \in \mathbb{N}$. Notice that $T_1(n)/T_2(n)$ is bounded above by some positive constant $C_1 > 0$ for $n < N$, since there are only finitely many such values of n. Additionally, since $[f_1(n)] = [f_2(n)]$, there exists a positive constant $C_2 > 0$ such that $f_1(n)/f_2(n)$ is bounded above by C_2.

Let us define $C = \max(C_1, C_2)$. Let us suppose that $T_1(n)/T_2(n) \leq C$, or $T_1(n) \leq CT_2(n)$, for some $n \in \mathbb{N}$, and suppose $m \in \mathbb{N}$ is such that $m \geq N$ and $bn \leq m < bn + b$, so that $\lfloor m/b \rfloor = n$. Then we have that

$$T_1(m) = aT_1(n) + f_1(n) \leq CaT_2(n) + C_2 f_2(n) \leq C(aT_2(n) + f_2(n)) \leq CT_2(m)$$

so that $T_1(m)/T_2(m)$ is again bounded above by C. Thus, we have that for all $m \geq N$, the ratio $T_1(m)/T_2(m)$ is bounded above by C if $T_1(\lfloor m/b \rfloor)/T_2(\lfloor m/b \rfloor)$ is bounded above by C. Since the ratio is bounded above by C for all $m < N$ by the definition of C, it follows by induction that $T_1(m)/T_2(m) \leq C$ for all $m \in \mathbb{N}$ and therefore $[T_1(n)] \leq [T_2(n)]$.

Hence we have that $[f_1(n)] \leq [f_2(n)] \implies [T_1(n)] \leq [T_2(n)]$, which is the second claim. Therefore, if $[f_1(n)] = [f_2(n)]$, so that both $[f_1(n)] \leq [f_2(n)]$ and $[f_1(n)] \geq [f_2(n)]$, we have that both $[T_1(n)] \leq [T_2(n)]$ and $[T_1(n)] \geq [T_2(n)]$, so that $[T_1(n)] = [T_2(n)]$, which proves the first claim. □

As a consequence of this proposition, the following definition is well-posed:

Definition 29. *If $a > 0, b > 1$ are constants and $\mathfrak{f}$ is a growth order, define* $\operatorname{divcon}(a, b, \mathfrak{f})$ *to be the growth order of any sequence $(T(n))$ satisfying*

$$T(n) = aT(\lfloor n/b \rfloor) + f(n)$$

for all sufficiently large n, where $(f(n)) \in \mathfrak{f}$.

We will shortly prove a proposition that greatly simplifies the special case of $\operatorname{divcon}(a, b, \mathfrak{f})$ with $a = 1$. First, however, we must prove the following lemma:

Proposition 82. *Suppose $T, T' : \mathbb{N} \to \mathbb{R}^+$ satisfy the following recurrences for all sufficiently large n:*

$$T(n) = T(a_n) + f(n)$$

$$T'(n) = T(b_n) + f(n)$$

where $f : \mathbb{N} \to \mathbb{R}^+$ is monotone increasing and (a_n), (b_n) are monotone increasing sequences of positive integers with $a_n \leq b_n < n$. Then we have that $[T(n)] \leq [T'(n)]$.

Proof. Let $N \in \mathbb{N}$ be such that T, T' satisfy these recurrences for all $n \geq N$. Without loss of generality, it suffices to prove the theorem for T, T' that are constant for all $n < N$, i.e. so that $T(n) = T'(n) = c > 0$ for all $n < N$. This is because the first $N - 1$ terms of T and T' will not affect their growth order.

First we will prove that T and T' are themselves monotone increasing functions. First of all, if $n \geq N$ and T is a monotone increasing function on $\{1, 2, \cdots, n\}$, we have that

$$T(n+1) = T(a_{n+1}) + f(n+1) \geq T(a_n) + f(n) = T(n)$$

since $a_{n+1} \geq a_n$ and $a_n, a_{n+1} \in \{1, 2, \cdots, n\}$, where T is assumed to be monotone. But since $T(n+1) \geq T(n)$, we have that T must also be monotone on the set $\{1, 2, \cdots, n+1\}$. We shall use this fact for an inductive argument. For the base case, notice that we know T is monotone on $\{1, 2, \cdots, N\}$ because $T(n) = c$ for $n = 1, 2, \cdots, N-1$ and $T(N) = c + f(N)$. Thus, by induction, we have that T is monotone on each set $\{1, 2, \cdots, n\}$ for all $n \geq N$, and therefore it is a monotone function $\mathbb{N} \to \mathbb{R}^+$. By similar reasoning, we also have that T' is monotone increasing.

Now we are ready to prove the desired claim. Suppose that, for some $n \geq N$, we know that $T(m) \leq T'(m)$ for all $m \in \{1, 2, \cdots, n\}$. Then we have that

$$T(n+1) = T(a_{n+1}) + f(n+1) \leq T'(a_{n+1}) + f(n+1) \leq T'(b_{n+1}) + f(n+1) = T'(n+1)$$

using the facts that $a_{n+1} \in \{1, 2, \cdots, n\}$, that $a_{n+1} \leq b_{n+1}$, and that T' is monotone increasing. Thus, the same inequality holds for all $m \in \{1, 2, \cdots, n+1\}$ as well. The inequality holds for all $m \in \{1, 2, \cdots, N\}$ clearly, since $T(n) = T'(n)$ for $n = 1, 2, \cdots, N$. Hence, we have that $T(m) \leq T'(m)$ for all $m \in \mathbb{N}$, and $[T(n)] \leq [T'(n)]$. □

Proposition 83. *For all $b > 1$ and moderate monotone $\mathfrak{f} \geq \mathbf{1}$, we have* $\mathrm{divcon}(1, b, \mathfrak{f}) = \mathrm{H}\mathfrak{f}$.

Proof. Clearly we have that $\mathrm{divcon}(1, 2, \mathfrak{f}) = \mathrm{H}\mathfrak{f}$ by the definition of H. Therefore, it suffices to show that the growth order $\mathrm{divcon}(1, b, \mathfrak{f})$ does not depend on the value of $b > 1$.

Let $b_1, b_2 > 1$ be given. Because $b_1 > 1$, there exists some exponent $p \in \mathbb{N}$ such that $b_1^p > b_2$. Now, if we consider a function $T : \mathbb{N} \to \mathbb{R}^+$ satisfying the recurrence

$$T(n) = T(\lfloor n/b_1 \rfloor) + f(n)$$

for all sufficiently large n for some monotone increasing function $f : \mathbb{N} \to \mathbb{R}^+$, then by repeatedly making substitutions in this recurrence, we have each of the following recurrences, each of which holds for sufficiently large n:

$$T(n) = T(\lfloor \lfloor n/b_1 \rfloor / b_1 \rfloor) + f(n) + f(\lfloor n/b_1 \rfloor)$$

$$T(n) = T(\lfloor \lfloor \lfloor n/b_1 \rfloor / b_1 \rfloor / b_1 \rfloor) + f(n) + f(\lfloor n/b_1 \rfloor) + f(\lfloor \lfloor n/b_1 \rfloor / b_1 \rfloor)$$

$$\cdots$$

Now, let us define a sequence of functions

$$f_1(n) = f(n)$$

$$f_2(n) = f(n) + f(\lfloor n/b_1 \rfloor)$$

$$\cdots$$

$$f_{k+1}(n) = f(n) + f_k(\lfloor n/b_1 \rfloor)$$

$$\cdots$$

Notice that each of the functions f_k is monotone increasing, since f is monotone increasing and the function $\lfloor n/b_1 \rfloor$ is monotone increasing. Notice also that $[f_k(n)] = [f(n)]$ for all $k \in \mathbb{N}$

by the moderateness of f. By the previous lemma, we may argue that $[T(n)] \leq [T'(n)]$, where T' is a function satisfying the following recurrence for all sufficiently large n:

$$T'(n) = T'(\lfloor n/b_1^p \rfloor) + f_p(n)$$

and, applying the lemma again and using the fact that $b_1^p \geq b_2$, we can argue that $[T'(n)] \leq [T''(n)]$, where T'' satisfies the following recurrence for all sufficiently large n:

$$T''(n) = T''(\lfloor n/b_2 \rfloor) + f_p(n)$$

Of course, the growth order of $(T''(n))$ is $\mathrm{divcon}(1, b_2, \mathfrak{f})$, since $[f_p(n)] = [f(n)]$. Thus, we have that $\mathrm{divcon}(1, b_1, \mathfrak{f}) \leq \mathrm{divcon}(1, b_2, \mathfrak{f})$ for arbitrary $b_1, b_2 > 1$. But since b_1, b_2 were arbitrary, we also have $\mathrm{divcon}(1, b_2, \mathfrak{f}) \leq \mathrm{divcon}(1, b_1, \mathfrak{f})$ and therefore $\mathrm{divcon}(1, b_1, \mathfrak{f}) = \mathrm{divcon}(1, b_2, \mathfrak{f})$, showing that the growth order $\mathrm{divcon}(1, b, \mathfrak{f})$ does not depend on the value of b. The desired claim follows. □

Proposition 84. *For all $a > 0$ and $b > 1$ and moderate monotone $\mathfrak{f} \geq \mathbf{1}$, we have that*

$$\mathrm{divcon}(a, b, \mathfrak{f}) = \mathfrak{n}^c \cdot \mathrm{divcon}(1, b, \mathfrak{n}^{-c} \cdot \mathfrak{f})$$

where $c = \log_b(a)$.

Proof. Let $f : \mathbb{N} \to \mathbb{R}^+$ be monotone increasing and consider the recurrence

$$T(n) = aT(\lfloor n/b \rfloor) + f(n)$$

If we multiply both sides of this recurrence by n^{-c}, where $c = \log_b(a)$, we obtain the equality

$$n^{-c}T(n) = (n/b)^{-c}T(\lfloor n/b \rfloor) + n^{-c}f(n)$$

Now, notice that

$$\frac{1}{(n/b)^c} = \frac{1}{(\lfloor n/b \rfloor + O(1))^c} = \frac{1}{\lfloor n/b \rfloor^c} - O\Big(\frac{1}{n^{c+1}}\Big)$$

which means that

$$n^{-c}T(n) = \lfloor n/b \rfloor^{-c}T(\lfloor n/b \rfloor) + n^{-c}f(n) - T(\lfloor n/b \rfloor) \cdot O\Big(\frac{1}{n^{c+1}}\Big)$$

Now, the monotonicity of f can be used to prove easily from the original recurrence for T that $[T(n)] \leq [n^c f(n)]$. This means that the quantity

$$T(\lfloor n/b \rfloor) \cdot O\Big(\frac{1}{n^{c+1}}\Big)$$

is also $n^{-c}f(n) \cdot O(n^{-1})$. We may use this bound to modify our recurrence:

$$n^{-c}T(n) = \lfloor n/b \rfloor^{-c}T(\lfloor n/b \rfloor) + n^{-c}f(n) \cdot (1 - O(n^{-1}))$$

Therefore, there exists a function $f' : \mathbb{N} \to \mathbb{R}^+$ such that $[f(n)] = [f'(n)]$ and, for all sufficiently large n,

$$n^{-c}T(n) = \lfloor n/b \rfloor^{-c} T(\lfloor n/b \rfloor) + n^{-c} f'(n)$$

Now, if we define a function $T' : \mathbb{N} \to \mathbb{R}^+$ by letting $T'(n) = n^{-c}T(n)$ for all $n \in \mathbb{N}$, then T' will satisfy the recurrence

$$T'(n) = T'(\lfloor n/b \rfloor) + n^{-c} f'(n)$$

and this function, of course, must have $[T'(n)] = \mathrm{divcon}(1, b, \mathfrak{n}^{-c} \cdot \mathfrak{f})$. But since $T'(n) = n^{-c}T(n)$, we have that $[T(n)] = \mathfrak{n}^c \cdot \mathrm{divcon}(1, b, \mathfrak{n}^{-c} \cdot \mathfrak{f})$ as claimed. □

Proposition 85. *Let (a_n) be a moderate and monotone sequence with growth order $\mathfrak{a} < \mathfrak{n}$ such that $a_n < n/2$ for all n sufficiently large. Additionally, let (b_n) be a moderate sequence with growth order $\mathfrak{b}$. If $f : \mathbb{N} \to \mathbb{R}$ satisfies the recurrence*

$$f(n) = f(n - a_n) + b_n$$

for all n sufficiently large, then the sequence $(f(n))$ has growth order $\mathrm{H}(\mathfrak{n}\mathfrak{b}/\mathfrak{a})$.

Proof. Let (a_n), (b_n) and $f(n)$ be defined as in the above statement. Because $a_n < n/2$ for all n sufficiently large and f satisfies the stated recurrence for all n sufficiently large, we may let 2^N be the smallest power of 2 such that $a_n < n/2$ and $f(n) = f(n - a_n) + b_n$ for all $n > 2^N$. Also, let M and M' be the respective maximum and minimum values among $f(1), \cdots, f(2^N)$. Additionally, since (b_n) is moderate, there exist constants $C_1 < 1 < C_2$ such that

$$C_1 b_n \leq b_m \leq C_2 b_n$$

for all $n \leq m \leq n$. We shall make use of the constants N, M, M', C_1, C_2 later in the proof.

We will use an induction-like argument to establish bounds on $f(n)$. Let us suppose that we have already proven the bound

$$g_p \leq f(n) \leq h_p$$

for all $n \in \{2^p + 1, \cdots, 2^{p+1}\}$, for some $p \geq N$. We will show that this can be used to obtain a similar bound for $f(n')$ for values of n' in the set $\{2^{p+1} + 1, \cdots, 2^{p+2}\}$. First of all, we have that $a_{n'} \geq a_{2^{p+1}}$ for all $n \in \{2^{p+1}+1, \cdots, 2^{p+2}\}$ by the monotonicity of α. Thus, by repeatedly applying the recurrence formula for f, we can repeatedly subtract quantities a_i from the argument of $f(n')$ and add quantities b_i to its value in order to express $f(n')$ in terms of $f(n)$ for some $n \in \{2^p + 1, \cdots, 2^{p+1}\}$. We will have to subtract quantities of the form a_i at most $\lceil 2^{p+1}/a_{2^{p+1}} \rceil$ many times, and the quantities that we subtract will be at most $C_2 b_{2^{p+1}}$ by the postulated bounds on β. This means that we have

$$f(n') \leq h_p + \left\lceil \frac{2^{p+1}}{a_{2^{p+1}}} \right\rceil \cdot C_2 b_{2^{p+1}}$$

Similarly, we have $a_{n'} \leq a_{2^{p+2}}$ for all $n' \in \{2^{p+1} + 1, \cdots, 2^{p+2}$, meaning that we apply the recurrence and subtract quantities a_i at least $\lfloor 2^{p+1}/a_{2^{p+2}} \rfloor$ times while still ending up with an argument n in the interval $n \in \{2^p + 1, \cdots, 2^{p+1}\}$. Each of the quantities b_i added will be at least $C_1^2 b_{2^p}$. Therefore, we have that

$$f(n') \geq g_p + \left\lfloor \frac{2^{p+1}}{a_{2^p}} \right\rfloor \cdot C_1^2 b_{2^p}$$

Therefore, if we define g_{p+1} and h_{p+1} as follows:

$$g_{p+1} = g_p + \left\lfloor \frac{2^{p+1}}{a_{2^p}} \right\rfloor \cdot C_1^2 b_{2^p}$$

$$h_{p+1} = h_p + \left\lceil \frac{2^{p+1}}{a_{2^{p+1}}} \right\rceil \cdot C_2 b_{2^{p+1}}$$

then we have that the inequalities $g_p \leq f(n) \leq h_p$ for all $n \in \{2^p + 1, \cdots, 2^{p+1}\}$ imply the inequalities $g_{p+1} \leq f(n') \leq h_{p+1}$ for all $n' \in \{2^{p+1} + 1, \cdots, 2^{p+2}\}$. By the definition of M and M', these bounds hold for the initial values of $g_{N-1} = M'$ and $h_{N-1} = M$. Thus, by induction, these inequalities hold for all $p \geq N - 1$ for the sequences (g_p), (h_p) defined by the above recurrences and the initial values $g_{N-1} = M'$ and $h_{N-1} = M$. Note that this implies that

$$g_{\lceil \log_2 n \rceil - 1} \leq f(n) \leq h_{\lceil \log_2 n \rceil - 1}$$

for all $n \geq 2^N$.

Now observe that, by the recursive definition of (g_p) and (h_p), we have that

$$g_p = M' + \sum_{k=N}^{p} \left\lfloor \frac{2^k}{a_{2^{k-1}}} \right\rfloor \cdot C_1^2 b_{2^{k-1}}$$

$$h_p = M + \sum_{k=N}^{p} \left\lceil \frac{2^k}{a_{2^k}} \right\rceil \cdot C_2 b_{2^k}$$

Now notice that the sequences are both equal to halving sums evaluated at the index 2^p of sequences with the same growth order as (nb_n/a_n), by the moderateness of (a_n) and (b_n). This means that, again using moderateness, the sequences $g_{\lceil \log_2 n \rceil - 1}$ and $h_{\lceil \log_2 n \rceil - 1}$ both have the growth order $\mathrm{H}(\mathfrak{n}\mathfrak{b}/\mathfrak{a})$. Therefore, since $f(n)$ is trapped beneath sequences of this growth order for all n sufficiently large, we have that the growth order of $(f(n))$ is also equal to $\mathrm{H}(\mathfrak{n}\mathfrak{b}/\mathfrak{a})$. $\square$

Let's see an example of this proposition in action. Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by the initial values $f(1) = f(2) = 1$ and the recurrence

$$f(n) = f\big(n - \lfloor \sqrt[3]{n} \rfloor\big) + \frac{1}{\log n}$$

Using the above proposition, we may let $\mathfrak{a} = \mathfrak{n}^{1/3}$ and $\mathfrak{b} = \mathfrak{l}^{-1}$ and instantly determine that the growth order of $f(n)$ is equal to $\mathrm{H}(\mathfrak{n}^{2/3}/\mathfrak{l}) = \mathfrak{n}^{2/3}/\mathfrak{l}$. In other words, the recursive formula for f gives rise to the following asymptotic behavior:

$$f(n) = \Theta\Big(\frac{n^{2/3}}{\log n}\Big)$$

7.3. Plateau sequences

THIS SECTION UNDER CONSTRUCTION!

In this section, we describe a construction that is useful for constructing inverses of monotone sequences. As usual, we start by defining a function on sequences, which we will later show is well-defined as a function on growth orders of sequences.

Definition 30. *Given a moderate sequence of positive integers $\alpha \in \mathcal{S}(\mathbb{N})$, define the* ***plateau sequence*** $\mathrm{plat}(\alpha)$ *to be the sequence consisting of a_1 ones followed by a_2 twos followed by a_3 threes and so on. That is, define entry n of the sequence* $\mathrm{plat}(\alpha)$ *to be the smallest positive integer P such that $a_1 + \cdots + a_P \geq n$.*

Proposition 86. *Whenever it is defined,* $\mathrm{plat}(\alpha)$ *is a moderate sequence.*

Proof. Let $\alpha = (a_n) \in \mathcal{S}(\mathbb{N})$ be a monotone increasing sequence and let $(P_n) = \mathrm{plat}(\alpha)$. Further let $n \leq m \leq kn$ for some positive integers $m, n, k \in \mathbb{N}$. By its definition, (P_n) is a monotone increasing sequence, so we have the lower bound

$$P_m \geq P_n$$

Now notice that, because (a_n) is moderate, there exists a constant C depending only on k such that $a_j \geq C a_n$ for all $n \leq j \leq kn$.

Now notice that, because (a_n) is a monotone increasing sequence, we have the inequality

$$a_1 + \cdots + a_{kP_n} \geq k(a_1 + \cdots + a_{P_n})$$

which proves that, since $a_1 + \cdots + a_{P_n} \geq n$ by definition, we also have $a_1 + \cdots + a_{kP_n} \geq kn \geq m$, and therefore

$$P_m \leq kP_n$$

Thus, in summary, we have

$$P_n \leq P_m \leq kP_n$$

which proves that $(P_n) = \mathrm{plat}(\alpha)$ is a moderate sequence. □

Next we prove that the growth order of $\mathrm{plat}(\alpha)$ depends only on the growth order of α.

Proposition 87. *If $\alpha, \alpha' \in \mathcal{S}(\mathbb{N})$ are monotone sequences such that $\alpha \sim \alpha'$, then* $\mathrm{plat}(\alpha) \sim \mathrm{plat}(\alpha)$.

7.4. Direct sum

Definition 31. *Given two monotone sequences $\alpha, \beta \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha], [\beta] > \mathbf{1}$, define their **direct sum** $\alpha \oplus \beta$ to be the sequence $\gamma = (c_n)$ defined as follows: if the set*

$$\{(a_i + b_j, i, j) \ : \ i, j \in \mathbb{N}\}$$

is endowed with the lexicographic ordering, then c_n is defined to be the first component of the nth-smallest element of this set.

First of all, it's not entirely obvious that this definition is valid at all. Why must the set

$$\{(a_i + b_j, i, j) \ : \ i, j \in \mathbb{N}\}$$

necessarily have an nth largest element? For pairs of sequences such as $a_n = n$ and $b_n = 1/n$, this definition does not work, because this set has no smallest element - however, these sequences do not satisfy the requirements of being monotone and $> \mathbf{1}$.

To see why this definition is valid, it suffices to show that for any $M \in \mathbb{R}^+$, there exist at most finitely pairs $(i, j) \in \mathbb{N}^2$ such that $a_i + b_j < M$. Since α, β are monotone and have growth orders $> \mathbf{1}$, they must be unbounded above, and there must therefore exist $N \in \mathbb{N}$ such that $a_n, b_n \geq M$ for all $n > N$. This means that $a_i + b_j \geq M$ for all (i, j) with either $i > N$ or $j > N$. Therefore, in order for $a_i + b_j < M$, we must have both i and j less than or equal to N. But there are only N^2 such pairs (i, j) satisfying this bound, meaning that there are only finitely many pairs for which $a_i + b_j < M$, and our definition is in fact admissible.

Proposition 88. *If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha \oplus \beta \sim \alpha' \oplus \beta'$.*

Proof. Suppose that $\alpha, \alpha', \beta, \beta'$ are monotone and have growth order $\geq \mathbf{1}$ such that $\alpha \sim \alpha'$ and $\beta \sim \beta'$. Then there exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$C_1 a'_n \leq a_n \leq C_2 a'_n$$

$$C_3 b'_n \leq b_n \leq C_4 b'_n$$

for all $n \in \mathbb{N}$. Let $\gamma = (c_n) = \alpha \oplus \beta$ and $\gamma' = (c'_n) = \alpha' \oplus \beta'$. Notice that, for all $i, j \in \mathbb{N}$, we have

$$a_i + b_j \leq C_2 a'_i + C_4 b'_j$$

and therefore

$$a_i + b_j \leq \max(C_2, C_4)(a'_i + b'_j)$$

Since the smallest n values of $a'_i + b'_j$ are all less than c'_n by the definition of γ, we have that $a_i + b_j$ is less than or equal to $\max(C_2, C_4)c'_n$ for each of the pairs (i, j) corresponding to the smallest n values of $a'_i + b'_j$. This means that there are at least n pairs (i, j) for which $a_i + b_j$ is less than or equal to $\max(C_2, C_4)c'_n$, implying that the nth smallest value of $a_i + b_j$ is under this value, and therefore

$$c_n \leq \max(C_2, C_4)c'_n$$

.

By similar reasoning, we may obtain the bound

$$c'_n \leq \max(C_1^{-1}, C_3^{-1})c_n$$

which, together with the previous bound, implies that

$$\max(C_1^{-1}, C_3^{-1})^{-1}c'_n \leq c_n \leq \max(C_2, C_4)c'_n$$

and therefore $\gamma \sim \gamma'$, proving the desired claim. □

This demonstrates that the growth order of the direct sum $\oplus$ of two sequences depends only on the growth orders of the two sequences. Therefore, we may define it elementwise on equivalence classes of sequences with the same growth order.

Definition 32. *For monotone growth orders $\mathfrak{a} = [\alpha], \mathfrak{b} = [\beta] \in \mathcal{S}(\mathbb{R}^+)/\sim$ which are both $> \mathbf{1}$, define their* ***direct sum*** *$\mathfrak{a} \oplus \mathfrak{b}$ as the growth order $[\alpha] \oplus [\beta] = [\alpha \oplus \beta]$.*

A. Lists for quick reference

A.1. List of counterexamples

Here's a list of pathological (but illuminating!) counterexamples appearing throughout the document:

1. Two incomparable growth orders:
   section 2.2, page 13
2. An uncountable antichain of incomparable growth orders:
   section 2.2, page 13
3. An increasing sequence of growth orders without a least upper bound:
   section 3.4, page 19
4. Two monotone yet incomparable growth orders:
   section 4.2, page 24
5. A sequence whose arithmetic subsequences have different growth order, in particular with (a_{2n}) growing faster than (a_n):
   section 1.3, page 10
6. A sequence whose translations have a different growth order, in particular with (a_{n+1}) growing faster than (a_n):
   section 1.3, page 10
7. A sequence that is bounded by polynomials yet immoderate:
   section 1.3, page 7
8. A sequence that is moderate yet incomparable to $\mathbf{1}$:
   section 4.2, page 24
9. Sequences demonstrating that subtraction of growth orders is ill-defined:
   section 3.3, page 16
10. Sequences demonstrating that exponentiation of growth orders is ill-defined:
    section 3.3, page 17
11. Sequences demonstrating the composite $\mathfrak{a} \circ \mathfrak{b}$ can be ill-defined if $\mathfrak{a}$ is immoderate:
    section 5.1, page 42
12. Two distinct incomparable growth orders with the same partial sum:
    section 4.1, page 22

13. Two distinct comparable growth orders with the same partial sum:
    section 4.1, page 22
14. Two growth orders $\mathfrak{a} < \mathfrak{b}$ such that $\mathfrak{a}/\Sigma\mathfrak{a} > \mathfrak{b}/\Sigma\mathfrak{b}$:
    section 4.3, page 31
15. A growth order which decays faster than each growth order of the form $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$, yet still has divergent partial sums:
    section 4.5, page 37
16. Growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ such that $\mathfrak{a} \geq 1$ and $\mathfrak{b} \leq \mathfrak{c}$ but $\mathfrak{a} \circ \mathfrak{b} \perp \mathfrak{a} \circ \mathfrak{c}$:
    section 5.2, page 44